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Linear stabilization of the programmed motions of non-linear controlled dynamical systems under parametric perturbations☆

Yu. K. Zotov

St. Petersburg, Russia

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ABSTRACT

A non-linear controlled dynamical system that describes the dynamics of a broad class of non-linear mechanical and electromechanical systems (in particular, electromechanical robot manipulators) is considered. It is proposed that the real parameter vector of a non-linear controlled dynamical system belongs to an assigned (admissible) constrained closed set and is assumed to be unknown. The programmed motion of the non-linear controlled dynamical system and the programmed control that produces it are assigned (constructed) by using an estimate, that is, the nominal value of the parameter vector of the non-linear controlled dynamical system, which differs from its actual value. A procedure for synthesizing stabilizing control laws with linear feedback with respect to the state that ensure stabilization of the programmed motions of the non-linear controlled dynamical system under parametric perturbations is proposed. A non-singular linear transformation of the coordinates of the state space that transforms the original non-linear controlled dynamical system in deviations (from the programmed motion and programmed control) into a certain non-linear controlled dynamical system of special form, which is convenient for analysing and synthesizing laws for controlling the motion of the system, is constructed. A certain non-linear controlled dynamical system of canonical form is derived in the original non-linear controlled dynamical system in deviations. The transformation of the coordinates of the state space constructed and the Lyapunov function methodology are used to synthesize stabilizing control laws with linear feedback with respect to the state, which ensure asymptotic stability as a whole of the equilibrium position of the non-linear controlled dynamical system of canonical form and dissipativity "in the large" of the non-linear controlled dynamical system of special form and of the original non-linear controlled dynamical system in deviations. In the control laws synthesized, the formulae for the elements of their matrices of the feedback loop gains do not depend on the real parameter vector of the non-linear controlled dynamical system, and they depend solely on the constants from certain estimates that hold for all of its possible values from an assigned set. Estimates of the region of dissipativity "in the large" of the non-linear controlled dynamical system of special form and the original non-linear controlled dynamical system in deviations closed by the stabilizing control laws synthesized are given, and estimates for their limit sets and regions of attraction are presented.

1. Statement of the problem

Consider a non-linear controlled dynamical system in the Cauchy problem of the form

$$\dot{z} = F(z, u, t, \xi), \quad z(t_0) = z_0, \quad t \geq t_0 \geq 0 \tag{1.1}$$

where z_0 and $z = z(t)$ are n-dimensional state vectors of the system at the initial and current times, u is an m-dimensional control vector, ξ is a p-dimensional parameter vector of the system,

$$\xi \in \Omega_\xi \subset R^p \tag{1.2}$$

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E-mail address: zyuko@mail.ru.

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Ω_ξ is a bounded closed set, R^p is a p-dimensional real Euclidean space, F is an n-dimensional vector function that satisfies (under an admissible control) the conditions for the existence and uniqueness of the solution of system (1.1) and determines the properties of a specific control object.

Let the programmed motion be assigned (constructed) in the form

$$z_p = z_p(t) \equiv z_p(t, \hat{\xi}), \quad t \geq t_0 \tag{1.3}$$

where

$$\hat{\xi} \in \Omega_\xi \tag{1.4}$$

is an estimate, that is, the nominal value of the parameter vector ξ of system (1.1), which is a particular solution of the system

$$\dot{z} = F(z, u, t, \hat{\xi}), \quad z(t_0) = z_0, \quad t \geq t_0 \geq 0 \tag{1.5}$$

This system is identical to system (1.1) for the value of the parameter vector

$$\xi = \hat{\xi} \tag{1.6}$$

and a certain admissible programmed control

$$u_p = u_p(t) \equiv u_p(t, \hat{\xi}), \quad t \geq t_0 \tag{1.7}$$

and the initial condition $z_0 = z_p(t_0) = z_{p0}$. The programmed motion $z_p(t)$ will be called the unperturbed motion, and any other motion $z(t)$ of system (1.1) under the admissible controls will be called the perturbed (real) motion.

The quantities

$$e = z - z_p, \quad e_u = u - u_p \tag{1.8}$$

are perturbations, i.e., deviations of the real (perturbed) motion z and the control u from their programmed values z_p and u_p. They are related by the differential equation in deviations

$$\dot{e} = F_e(e, e_u, t, e_\xi), \quad e(t_0) = e_0, \quad t \geq t_0 \tag{1.9}$$

where

$$e_\xi = \xi - \hat{\xi} \ (e_\xi \in \Omega_{e\xi}) \tag{1.10}$$

is a parametric perturbation, i.e., a deviation of the real parameter vector ξ from its nominal value $\hat{\xi}$, and the set

$$\Omega_{e\xi} = \{e_\xi \in R^p: e_\xi + \hat{\xi} \in \Omega_\xi\} \tag{1.11}$$

$$F_e(e, e_u, t, e_\xi) = F(e + z_p, e_u + u_p, t, e_\xi + \hat{\xi}) - F(z_p, u_p, t, \hat{\xi}) \tag{1.12}$$

where $F_e(0, 0, t, 0) \equiv 0$. It follows from equality (1.12) that under the controls $e_u = 0$ and $e_\xi = 0$ system (1.9)–(1.12) has the motion $e \equiv 0$.

The transformations (1.8) reduce the problem of studying the motions $z(t)$ of the original non-linear controlled dynamical system (1.1) in the neighbourhood of any isolated programmed motion $z_p(t)$ to the problem of studying the solutions $e = e(t)$ of the original non-linear controlled dynamical system in deviations (1.9)–(1.12) in the neighbourhood of the origin of coordinates $e = 0$; therefore, in the ensuing discussion the main constraints, assumptions, and assertions will be formulated with reference to the original non-linear controlled dynamical system in deviations (1.9)–(1.12).

For a broad class of mechanical and electromechanical systems, the structure of the original non-linear controlled dynamical system in deviations (1.9)–(1.12) is such that

$$\dot{e} = P_0(e^{r-2}, t, \xi)e + Q_0(e^{r-1}, t, \xi)e_u + g_e(e, t, e_\xi), \quad e(t_0) = e_0, \quad t \geq t_0 \tag{1.13}$$

Here $e = \mathrm{col}(e_1, \ldots, e_r)$, e_i and $e^i = \mathrm{col}(e_1, \ldots, e_r)$ are n, m and mi vectors, $rm = n$, $\xi = e\xi + \hat{\xi}$, and

$$P_0(e^{r-2}, t, \xi)e + Q_0(e^{r-1}, t, \xi)e_u + g_e(e, t, e_\xi) \equiv F_e(e, e_u, t, e_\xi) \tag{1.14}$$

where

$$P_0(e^{r-2}, t, \xi) = \left\| \begin{matrix} 0 & P_{012}(t, \xi) & 0 & \ldots & \ldots & 0 \\ 0 & 0 & P_{023}(e^1, t, \xi) & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & \ldots & 0 & P_{0,r-1,r}(e^{r-2}, t, \xi) \\ 0 & 0 & \ldots & \ldots & 0 & 0 \end{matrix} \right\| \tag{1.15}$$

$$Q_0(e^{r-1}, t, \xi) = \left\| \begin{matrix} 0 \\ P_{0r, r+1}(e^{r-1}, t, \xi) \end{matrix} \right\| \tag{1.16}$$

are partitioned matrix functions of order $n \times n$ and $n \times m$, respectively, and $P_{0k,k+1}$ $(k = 1, \ldots, r)$ are $m \times m$ partitioned matrix functions, which can be represented in the form

$$P_{012}(t, \xi) = A_1(t, \xi)B_1(t, \xi), \quad P_{0k, k+1}(e^{k-1}, t, \xi) = A_k(e^{k-1}, t, \xi)B_k(t, \xi),$$
$$k = 2, \ldots, r \tag{1.17}$$

where

$$A_1(t, \xi) = A_1^*(t, \xi) > 0, \quad t \geq t_0, \quad \xi \in \Omega_\xi$$
$$A_k(e^{k-1}, t, \xi) = A_k^*(e^{k-1}, t, \xi) > 0, \quad (e^{k-1}, t, \xi) \in \Omega_{1, k-1}, \quad k = 2, \ldots, r \tag{1.18}$$
$$B_k(t, \xi) = B_k^*(t, \xi) > 0, \quad t \geq t_0, \quad \xi \in \Omega_\xi, \quad k = 1, \ldots, r \tag{1.19}$$

are symmetrical, positive definite $m \times m$ matrix functions, such that

$$|A_1(t, \xi)| \leq k_{A1}, \quad t \geq t_0, \quad \xi \in \Omega_\xi; \quad \left|A_k(e^{k-1}, t, \xi)\right| \leq k_{Ak}, \quad (e^{k-1}, t, \xi) \in \Omega_{1, k-1},$$
$$k = 2, \ldots, r \tag{1.20}$$
$$|B_k(t, \xi)| \leq k_{Bk}, \quad |\dot{B}_k(t, \xi)| \leq \bar{k}_{Bk}, \quad t \geq t_0, \quad \xi \in \Omega_\xi, \quad k = 1, \ldots, r \tag{1.21}$$

$0 < k_{Ak} < \infty$, $0 < k_{Bk} < \infty$ and $0 \leq \bar{k}_{Bk} < \infty (k = 1, \ldots, r)$ are certain constants; similar estimates exist for the partial derivatives of the matrix functions A_k and B_k $(k = 1, \ldots, r)$ with respect to their arguments t and e^{k-1}; the set is

$$\Omega_{1, k-1} = \{(e^{k-1}, t, \xi) : e^{k-1} \in R^{m(k-1)}, t \geq t_0, \xi \in \Omega_\xi\} \tag{1.22}$$

Here the asterisk denotes the operation of transposition, and 0 is the zero matrix of the respective dimension.

When relations (1.12) and (1.14) are taken into account, the vector function $g_e(e, t, e_\xi)$ can be represented in the form

$$g_e(e, t, e_\xi) = \mathrm{col}(g_{e1}(e^1, t, e_\xi), g_{e2}(e^2, t, e_\xi), \ldots, g_{er}(e^r, t, e_\xi)) = \bar{g}_e(e, t, e_\xi + \hat{\xi}) + \hat{F}_e(t, e_\xi)$$
$$(F_e(e, e_u, t, e_\xi) = \bar{F}_e(e, e_u, t, e_\xi + \hat{\xi}) + \hat{F}_e(t, e_\zeta) = P_0(e^{r-2}, t, \xi)e + Q_0(e^{r-1}, t, \xi)e_u +$$
$$+ g_e(e, t, e_\xi),$$
$$\bar{F}_e(e, e_u, t, e_\xi + \hat{\xi}) = F(e + z_p, e_u + u_p, t, e_\zeta + \hat{\xi}) - F(z_p, u_p, t, e_\xi + \hat{\xi}) =$$
$$= P_0(e^{r-2}, t, e_\xi + \hat{\xi})e + Q_0(e^{r-1}, t, e_\xi + \hat{\xi})e_u + \bar{g}_e(e, t, e_\xi + \hat{\xi}),$$
$$\bar{F}_e(0, 0, t, e_\xi + \hat{\xi}) = 0, \quad \hat{F}(t, e_\xi) = F(z_p, u_p, t, e_\xi + \hat{\xi}) - F(z_p, u_p, t, \hat{\xi}), \quad \hat{F}_e(t, 0) = 0) \tag{1.23}$$

where the g_{ek} $(k = 1, \ldots r)$ are m-vector functions. The vector function g_e (1.23) satisfies the estimate

$$|g_e(e, t, e_\xi)| = \left|\bar{g}_e(e, t, e_\xi + \hat{\xi}) + \hat{F}_e(t, e_\xi)\right| \leq k_{ge0} + k_{ge1}|e| + k_{ge2}|e|^2, \ (e, t, e_\zeta) \in \Omega_{2, r}$$
$$(\left|\bar{g}_e(e, t, e_\xi + \hat{\xi})\right| \leq k_{ge1}|e| + k_{ge2}|e|^2, \ (e, t, e_\xi) \in \Omega_{2, r}, \ \left|\hat{F}_e(t, e_\xi)\right| \leq k_{ge0}, \ t \geq t_0, \ e_\xi \in \Omega_{e\xi}) \tag{1.24}$$

Here

$$\Omega_{2, r} = \{(e^r, t, e_\xi) : e^r = e \in R^n, t \geq t_0, e_\xi \in \Omega_{e\xi}\} \tag{1.25}$$
$$0 < k_{ge0} < \infty, \quad 0 \leq k_{ge1} < \infty, \quad 0 < k_{ge2} < \infty \tag{1.26}$$

the k_{gel} $(l = 0, 1, 2)$ are certain constants, $|a| = (a_1^2 + \ldots + a_n^2)^{1/2}$, and $|A| = \left(\sum_{i=1}^n \sum_{j=1}^m a_{ij}^2\right)^{1/2}$ are the moduli (Euclidean norms) of the real vector $a = \mathrm{col}(a_1, \ldots, a_n) \in R^n$ and the real matrix $A = ||a_{ij}||_{i=1,\ldots,n; j=1,\ldots,m}$ of order $n \times m$.

Below the control law u for the original non-linear controlled dynamical system (1.1), in which the vector function F is such that relations (1.12) and (1.14)–(1.26) hold, has the structure of a control law with linear feedback with respect to the state z of the form

$$u = u(z, t, \hat{\xi}) = u_p(t, \hat{\xi}) + \Gamma_0(z - z_p), \quad t \geq t_0 \tag{1.27}$$

where

$$\Gamma_0 = \|\Gamma_{01}, \ldots, \Gamma_{0r}\| \tag{1.28}$$

is a constant $m \times n$ partitioned matrix of feedback loop gain and the Γ_{0k} ($k = 1, \ldots, r$) are $m \times m$ blocks. The control law for the original non-linear controlled dynamical system in deviations (1.13)–(1.26) has the structure of a control law with linear feedback with respect to the state e and is written accordingly in the form

$$e_u = \Gamma_0 e \tag{1.29}$$

We will say that for the original non-linear controlled dynamical system in deviations (1.13)–(1.26) the origin of coordinates $e = 0$ can be stabilized by the control law e_u (1.29), (1.28) with linear feedback with respect to the state vector $e(t)$, if this control law ensures dissipativity "in the large" at the origin of coordinates $e = 0$ of the original closed non-linear controlled dynamical system in deviations (1.13)–(1.26), (1.29), (1.28):

$$\dot{e} = P_0(e^{r-2}, t, \xi)e + Q_0(e^{r-1}, t, \xi)\Gamma_0 e + g_e(e, t, e_\xi), \quad e(t_0) = e_0, \quad t \ge t_0 \tag{1.30}$$

Accordingly, for the original non-linear controlled dynamical system (1.1), (1.12), (1.14)–(1.26), the programmed motion $z_p(t)$ (1.3) can be stabilized by the control law u (1.27), (1.28) with linear feedback with respect to the state vector $z(t)$ if this control law ensures dissipativity "in the large" of the programmed motion $z_p(t)$ (1.3) of the closed original non-linear controlled dynamical system (1.1), (1.12), (1.14)–(1.26), (1.27), (1.28):

$$\dot{z} = F(\dot{z}, u_p(t, \hat{\xi}) + \Gamma_0(z - z_p), t, \xi), \quad z(t_0) = z_0, \quad t \ge t_0 \tag{1.31}$$

according to the definition given below.

Definition 1. (Ref. 1, p. 126). The set Ω_0 is called a region of attraction of system (1.30) if

1) for any solution $e(t)$ a time t_0 is found such that $e(t_0) \in \Omega_0$;
2) the set Ω_0 is invariant, i.e., it follows from $e(t_0) \in \Omega_0$ that $e(t;\, \mathrm{e}(t_0), t_0) \in \Omega_0$ at all $t \ge t_0$.

Definition 2. (Ref. 1, p. 126). System (1.30) is called a dissipative system if a bounded closed region of attraction Ω_0 exists in the n-dimensional space $\{e\}$.

Definition 3. System (1.30) will be called a system that is dissipative in the large if each trajectory $e(t;\, e_0, t_0)$ that emerges from a certain bounded closed set $\Omega_0 \subset R^n$ ($e_0 \in \Omega_0$) enters a certain closed neighbourhood $\Omega_1 \subset \Omega_0$ of the origin of coordinates $e = 0 \in \Omega_1$ at a sufficiently long time $t = t_* \ge t_0$ and does not leave it afterwards (i.e., at $t \ge t_*$). We will call such a neighbourhood Ω_1 the limit set of the dissipative system.

Otherwise, for each solution $e(t;\, e_0, t_0)$ ($e_0 \in \Omega_0 \subset R^n$) there is a time $t_* = t_0 + T(t_0, e_0) \ge t_0$, after which the solution sinks everywhere into the fixed sphere $\Omega_1 = \{e \in R^n : |e| \le R_0\} \subset \Omega_0$, i.e.,

$$|e(t;\, t_0, e_0)| < R_0 \text{ at } t_* \le t < \infty$$

We next formulate the stabilizability criteria of the origin of coordinates $e = 0$ for the original non-linear controlled dynamical system in deviations (1.13)–(1.26) with the control law e_u (1.29), (1.28) with linear feedback with respect to the state e (and the programmed motion $z_p(t)$ (1.3) for the original non-linear controlled dynamical system (1.1), (1.12), (1.14)–(1.26) with the control law u (1.27), (1.28) with linear feedback with respect to z, respectively). Estimates of the regions of dissipativity "in the large" of the closed original non-linear controlled dynamical system in deviations (1.13)–(1.26), (1.29), (1.28), i.e., system (1.30) (and of the closed original non-linear controlled dynamical system (1.1), (1.12), (1.14)–(1.26), i.e., system (1.31), respectively) are given, and estimates are presented for its limit set and region of attraction.

Similarly stated stabilization problems of controlled dynamical systems under parametric perturbations were previously considered.[2–8]

2. Reductions of the original non-linear controlled dynamical system in deviations to a non-linear controlled dynamical system of special form

The procedure proposed below for the parametric synthesis of a stabilizing control law with linear feedback with respect to the state of the non-linear controlled dynamical system in deviations (1.13)–(1.26) and the analysis of the behaviour of the solutions in a closed system involves reducing this system to a certain non-linear controlled dynamical system of special form, which is more convenient for examining these questions.

For this purpose, in the original non-linear controlled dynamical system in deviations (1.13)–(1.26) we perform a non-singular linear transformation of the coordinates of the state space of the form

$$e_x = Se, \quad e = S^{-1}e_x = Re_x \tag{2.1}$$

Here

$$e_x = \mathrm{col}(e_{x1}, \ldots, e_{xr}) \tag{2.2}$$

and $e_{xk} = \mathrm{col}(e_{xk1}, \ldots, e_{xkm})$ are n- and m-dimensional vectors, and S and R are non-singular constant lower triangular $n \times n$ partitioned matrices of the form

$$S = \begin{Vmatrix} I_m & 0 & \ldots & & \ldots & 0 \\ S_{21} & I_m & 0 & \ldots & \ldots & 0 \\ S_{32}S_{21} & S_{32} & I_m & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ S_{r-1,r-2}S_{r-2,r-3}\ldots S_{21} & S_{r-1,r-2}S_{r-2,r-3}\ldots S_{32} & \ldots & S_{r-1,r-2} & I_m & 0 \\ S_{r,r-1}S_{r-1,r-2}\ldots S_{21} & S_{r,r-1}S_{r-1,r-2}\ldots S_{32} & \ldots & \ldots & S_{r,r-1} & I_m \end{Vmatrix}$$
$$= \|S_{kl}\|_{k,l=1,\ldots,r} \tag{2.3}$$

where I_m is an $m \times m$ identity matrix

$$S_{kl} = 0, \quad k = 1, \ldots, r-1; \quad l = k+1, \ldots, r; \quad S_{kk} = I_m, \quad k = 1, \ldots, r$$
$$S_{kl} = S_{k,k-1}S_{k-1,l} = S_{k,k-1}S_{k-1,k-2}\ldots S_{l+1,l}, \quad k = 3, \ldots, r; \quad l = 1, \ldots, k-2 \tag{2.4}$$

the $S_{k+1,k}$ ($k = 1, \ldots, r-1$) are $m \times m$ blocks, whose analytic form is indicated below in Section 4 (in Lemma 1),

$$R = S^{-1} = \|R_{kl}\|_{k,l=1,\ldots,r}$$
$$R_{kl} = 0,\ k = 1, \ldots, r-1;\ l = k+1, \ldots, r;\ R_{kl} = 0,\ k = 3, \ldots, r;\ l = 1, \ldots, k-2$$
$$R_{kk} = I_m, \quad k = 1, \ldots, r; \quad R_{k,k-1} = -S_{k,k-1}, \quad k = 2, \ldots, r \tag{2.5}$$

Then the original non-linear controlled dynamical system in deviations (1.13)–(1.26) is transformed into a non-linear controlled dynamical system of special form

$$\dot{e}_x = P(e_x^{r-2}, t, \xi)e_x + Q(r_x^{r-1}, t, \xi)e_u + g_{ex}(e_x, t, e_\xi), \ e_x(t_0) = e_{x0}, \ t \geq t_0; \ \xi = e_\xi + \hat{\xi} \tag{2.6}$$

Here

$$P(e_x^{r-2}, t, \xi)e_x + Q(e_x^{r-1}, t, \xi)e_u + g_{ex}(e_x, t, e_\xi) \equiv F_{ex}(e_x, e_u, t, e_\xi) = SF_e(Re_x, e_u, t, e_\xi) \tag{2.7}$$

F_e is the vector function (1.14)–(1.26), and

$$P(e_x^{r-2}, t, \xi) = S[P_0(\sigma^{r-2}(e_x^{r-2}), t, \xi)]R = \|P_{kl}\|_{k,l=1,\ldots,r} \tag{2.8}$$

is an $n \times n$ partitioned matrix function, and its $m \times m$ blocks have the form

$$P_{11} \equiv P_{11}(t, \xi) = -P_{012}(t, \xi)S_{21}, \quad P_{12} \equiv P_{12}(t, \xi) = P_{012}(t, \xi)$$
$$P_{k1} \equiv P_{k1}(t, \xi) = -S_{k1}P_{012}(t, \xi)S_{21}, \quad k = 2, \ldots, r$$
$$P_{k,k+1} \equiv P_{k,k+1}(e_x^{k-1}, t, \xi) = P_{0,k,k+1}(\sigma^{k-1}(e_x^{k-1}), t, \xi), \quad k = 2, \ldots, r-1$$
$$P_{kk} \equiv P_{kk}(e_x^{k-1}, t, \xi) = S_{k,k-1}P_{0,k-1,k}(\sigma^{k-2}(e_x^{k-2}), t, \xi)$$
$$-P_{0,k,k+1}(\sigma^{k-1}(e_x^{k-1}), t, \xi)S_{k+1,k}, \quad k = 2, \ldots, r-1$$
$$P_{kl} = 0, \quad k = 1, \ldots, r-2; \quad l = k+2, \ldots, r$$
$$P_{kl} \equiv P_{kl}(e_x^{l-1}, t, \xi) = S_{k,l-1}P_{0,l-1,l}(\sigma^{l-2}(e_x^{l-2}), t, \xi)$$
$$-S_{kl}P_{0,l,l+1}(\sigma^{l-1}(e_x^{l-1}), t, \xi)S_{l+1,l}, \quad k = 3, \ldots, r; \quad l = 2, \ldots, k-1 \tag{2.9}$$
$$P_{rr} \equiv P_{rr}(e_x^{r-2}, t, \xi) = S_{r,r-1}P_{0,r-1,r}(\sigma^{r-2}(e_x^{r-2}), t, \xi)$$

$e_x^k = \mathrm{col}(e_{x1}, \ldots, e_{xk})$, $e_{xk} = \mathrm{col}(e_{xkl}, \ldots e_{xkm})$. Here and everywhere below

$$\sigma^k \equiv \sigma^k(e_x^k) = \mathrm{col}(\sigma_1(e_{x1}), \sigma_2(e_{x1}, e_{x2}), \ldots, \sigma_k(e_{x,k-1}, e_{xk}))$$
$$= H_kRe_x = H_ke = e^k = \mathrm{col}(e_1, \ldots, e_k) \tag{2.10}$$

$(\sigma_1(e_{x1}) = e_{x1} = e_1, \sigma_k(e_{x,k-1}, e_{xk}) = -S_{k,\,k-1}e_{x,\,k-1} + I_m e_{xk} = e_k, k = 2, \ldots, r)$ is an mk-dimensional vector function, where $H_k = ||I_{km}, 0||$ is a constant partitioned matrix of order $(km) \times n$, and everywhere

$$P_{012}(\sigma^0(e_x^0), t, \xi) \equiv P_{012}(t, \xi) \tag{2.11}$$

is an $m \times m$ block. When (1.16), (2.3) and (2.4) are taken into account, Q becomes a partitioned matrix function of the form

$$Q(e_x^{r-1}, t, \xi) = SQ_0(\sigma^{r-1}(e_x^{r-1}), t, \xi) = Q_0(\sigma^{r-1}(e_x^{r-1}), t, \xi) \tag{2.12}$$

$$g_{ex}(e_x, t, e_\xi) = Sg_e(Re_x, t, e_\xi) \tag{2.13}$$

is an n-vector function, for which the estimate

$$|g_{ex}(e_x, t, e_\xi)| = |Sg_e(Re_x, t, e_\xi)| \le |S||g_e(Re_x, t, e_\xi)|$$
$$\le |S|(k_{ge0} + k_{ge1}|Re_x| + k_{ge2}|Re_x|^2) \le k_{gex0} + k_{gex1}|e_x| + k_{gex2}|e_x|^2, \quad (e_x, t, e_\xi) \in \Omega_{2x,\,r} \tag{2.14}$$

where

$$\Omega_{2x,\,r} = \{(e_x^r, t, e_\xi) : e_x^r \in R^n, t \ge t_0, e_\xi \in \Omega_{e\xi}\} \tag{2.15}$$

$$0 < k_{gex0} = |S|k_{ge0} < \infty, \quad 0 \le k_{gex1} = |S||R|k_{ge1} < \infty, \quad 0 < k_{gex2} = |S||R|^2 k_{ge2} < \infty \tag{2.16}$$

and k_{gexj}, k_{gej} $(j = 0, 1, 2)$ are certain constants, holds when (1.23)–(1.26) and (2.1)–(2.5) are taken into account.

Note that since the matrix functions P_0 (1.15), (1.17)–(1.22) and Q_0 (1.16) have canonical forms in the original non-linear controlled dynamical system in deviations (1.13)–(1.26), it is possible to construct a non-singular linear transformation of the coordinates of the state space (2.1)–(2.4) that transforms this system into the non-linear controlled dynamical system of special form (2.6)–(2.16).

3. An auxiliary lemma regarding the dissipativity “in the large” of a non-linear dynamical system

Let us consider the non-linear dynamical system

$$\dot{e} = f(e, t) + g(e, t), \quad e(t_0) = e_0, \quad t \ge t_0 \ge 0 \tag{3.1}$$

where $e_0 = e(t_0) = e(t_0; e_0, t_0)$ and $e = e(t) = e(t; e_0, t_0)$ are n-dimensional state vectors of the system at the initial and current times, and f and g are continuous n-dimensional vector functions, for which $f(0, t) \equiv 0$ and

$$|g(e, t)| \le k_{g0} + k_{g1}|e| + k_{g2}|e|^2, \quad \forall e \in R^n, \quad \forall t \ge t_0$$
$$0 < k_{g0} < \infty, \quad 0 \le k_{g1} < \infty, \quad 0 < k_{g2} < \infty \tag{3.2}$$

It is assumed that for system (3.1), (3.2) the solution of the Cauchy problem exists and is unique.

Lyapunov function methodology enables us to find effective estimates of the dimensions of the limit set and the region of attraction (Ref. 1, Ref. 9, pp. 29–60, Ref. 10, pp. 289–293 and Ref. 11) of the dissipative system.

Thus, following this methodology (Ref. 11, p. 150), we will assume that $v(e, t)$ is a real, continuously differentiable positive definite scalar function (a Lyapunov function), $v(e, t) = 0$, and the regions

$$\Omega_0 = \{e \in R^n : \upsilon(e, t) \le \rho_{\upsilon 0}, \rho_{\upsilon 0} > 0, t \ge t_0\} \tag{3.3}$$

$$\Omega_1 = \{e \in R^n : \upsilon(e, t) \le \rho_{\upsilon 1}, \rho_{\upsilon 1} > 0, t \ge t_0\} \tag{3.4}$$

(with a centre at the origin of coordinates $e = 0$) are such that

$$\Omega_1 \subset \Omega_0, \text{ если } 0 < \rho_{\upsilon 1} < \rho_{\upsilon 0} \tag{3.5}$$

and the function $\dot{v}(e, t)$ along the trajectory of system (3.1), (3.2) satisfies the estimate

$$\dot{\upsilon} = \dot{\upsilon}(e(t), t) = \frac{\partial \upsilon}{\partial t} + \frac{\partial \upsilon}{\partial e}(f(e, t) + g(e, t)) = w(e(t), t) = w(e, t) \le -w_0(e) < 0$$

(where $w_0(e)$ is a positive-definite scalar function and $w_0(e) = 0$ in the layer

$$\Omega_D = \Omega_0 \backslash \Omega_1 \tag{3.6}$$

Then all the trajectories of system (3.1), (3.2) that begin at $t = t_0$ in the region Ω_D $(e(t_0) \in \Omega_D)$ enter the region Ω_1 at a certain sufficiently long time $t = t_*$ (in the general case, the value of t_* is different for different trajectories) and do not leave this region afterwards (i.e., for all $t \ge t_*$), i.e.,

$$e(t; e(t_*), t_*) \in \Omega_1, \quad \forall t \ge t_* \ge t_0, \quad e(t_0) \in \Omega_D \tag{3.7}$$

In such a case we will say (Ref. 11, p. 151) that system (3.1), (3.2) is dissipative in the large and that the regions Ω_1 (3.4), (3.5) and Ω_0 (3.3) serve as estimates of the limit set and the region of attraction of system (3.1), (3.2).

Auxiliary lemma. *Suppose a real scalar function $\upsilon \equiv \upsilon(e, t)$ exists, which is continuously differentiable with respect to its arguments, except for the argument $e = 0$, and let there be the real numbers $\varepsilon_{\upsilon i} > 0$ ($i = 1, 2, 3$), $\alpha_0 > 0$ and $0 < \nu_0 < 1$ such that*

1) $\varepsilon_{\upsilon 1}|e| \le \upsilon(e, t) \le \varepsilon_{\upsilon 2}|e|, \forall e \in R^n, t \ge t_0, \upsilon(0, t) = 0$
2) $\left|\frac{\partial \upsilon}{\partial e}\right| \le \varepsilon_{\upsilon 3}, \left|\frac{\partial \upsilon}{\partial e}\right| \ne 0, |e| \ne 0$
3) *in estimate (3.2) for the vector function $g(e, t)$, the coefficients k_{gj} ($j = 0, 1, 2$) are such that*

$$0 < k_{g0} < \infty, \quad 0 \le k_{g1} < (1 - \nu_0)\alpha_0\varepsilon_{\upsilon 1}\varepsilon_{\upsilon 3}^{-1}, \quad 0 < \nu_0 < 1$$
$$0 < k_{g2} < \infty, \quad [k_{g1} - (1 - \nu_0)\alpha_0\varepsilon_{\upsilon 1}\varepsilon_{\upsilon 3}^{-1}]^2 - 4k_{g2}k_{g0} > 0 \tag{3.8}$$

4) *by virtue of the system*

$$\dot{e} = f(e, t), \quad e(t_0) = e_0, \quad t \ge t_0 \tag{3.9}$$

the derivative of the function $\upsilon(e(t), t)$, along the non-trivial solution $e(t) = e(t; e_0, t_0)$ of this system satisfies the estimate

$$\frac{d}{dt}\upsilon(e(t), t) = \frac{\partial \upsilon}{\partial t} + \frac{\partial \upsilon}{\partial e}f(e(t), t) \le -\alpha_0\upsilon(e(t), t), \quad t \ge t_0 \tag{3.10}$$

Then,

1) system (3.1), (3.2), (3.8) is dissipative "in the large", and the regions

$$\Omega_0 = \{e \in R^n : \upsilon(e, t) < \rho_{\upsilon 0}, t \ge t_0\} \tag{3.11}$$
$$\Omega_1 = \{e \in R^n : \upsilon(e, t) < \rho_{\upsilon 1}, t \ge t_0\}, \quad \Omega_1 \subset \Omega_0 \tag{3.12}$$

where the real numbers are given by the formulae

$$\rho_{\upsilon 0} = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \rho_{\upsilon 1} = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \quad \rho_{\upsilon 0} > \rho_{\upsilon 1} > 0$$
$$(a = \varepsilon_{\upsilon 3}\varepsilon_{\upsilon 1}^{-2}k_{g2}, \ b = \varepsilon_{\upsilon 3}\varepsilon_{\upsilon 1}^{-1}[k_{g1} - (1 - \nu_0)\alpha_0\varepsilon_{\upsilon 1}\varepsilon_{\upsilon 3}^{-1}] < 0, \ c = \varepsilon_{\upsilon 3}k_{g0}) \tag{3.13}$$

are estimates of the region of attraction Ω_0 (3.3) and the limit set Ω_1 (3.4), (3.5) of this system, respectively;
2) in the region

$$\Omega_D = \Omega_0 \backslash \Omega_1 \tag{3.14}$$

where Ω_0 and Ω_1 are the sets (3.11) and (3.12), respectively, and the solution $e(t)$ of system (3.1), (3.2), (3.8) satisfies the estimate

$$|e(t)| \le \beta_0 e^{-\gamma_0(t - t_0)}|e(t_0)|, \quad e(t_0) \in \Omega_D, \quad t_0 \le t \le t_*; \quad \beta_0 = \varepsilon_{\upsilon 2}\varepsilon_{\upsilon 1}^{-1}, \quad \gamma_0 = \nu_0\alpha_0 \tag{3.15}$$

where t_* is a certain time, which is such that

$$e(t_*; e_0, t_0) \in \Omega_1, \quad e(t; e(t_*), t_*) \in \Omega_1, \quad \forall t \ge t_* \ge t_0 \tag{3.16}$$

Proof. Taking into account conditions 1–4 of the lemma, we calculate the derivative of the function $\upsilon \equiv \upsilon(e(t), t)$ with respect to the time t by virtue of system (3.1), (3.2), (3.8). We obtain

$$\dot{\upsilon} = \frac{\partial \upsilon}{\partial t} + \frac{\partial \upsilon}{\partial e}(f(e, t) + g(e, t)) \le -\alpha_0\upsilon + \left|\frac{\partial \upsilon}{\partial e}\right||g(e, t)| \le -\alpha_0\upsilon + \varepsilon_{\upsilon 3}(k_{g0} + k_{g1}|e| + k_{g2}|e|^2) \le$$
$$\le -\gamma_0\upsilon + a\upsilon^2 + b\upsilon + c = -\gamma_0\upsilon + a(\upsilon - \rho_{\upsilon 0})(\upsilon - \rho_{\upsilon 1}) \le -\gamma_0\upsilon, \quad e(t) \in \Omega_D, \quad t_0 \le t \le t_*; \quad \gamma_0 = \nu_0\alpha_0 \tag{3.17}$$

where a, b, c, $\rho_{\upsilon 0}$ and $\rho_{\upsilon 1}$ are the real numbers (3.13), $b^2 - 4ac > 0$ when the last relation in (3.8) is taken into account, Ω_D is the set (3.14), (3.11)–(3.13), and $t_* \ge t_0$ is a certain time such that $e(t_*; e_0, t_0) \in \Omega_1$.

It follows from estimate (3.17) and condition 1 of the lemma that the inequalities

$$\upsilon(e, t) \le e^{-\gamma_0(t - t_0)}\upsilon(e_0, t_0), \quad |e(t)| \le \beta_0 e^{-\gamma_0(t - t_0)}|e_0|, \quad e_0 \in \Omega_D, \quad t_0 \le t \le t_*$$

hold, where

$$e(t) = e(t, e(t_*), t_*) \in \Omega_1, \quad t \ge t_*$$

Consequently, the assertions of the lemma hold. □

4. Criteria for the linear stabilizability of a non-linear controlled dynamical system

1°. We will first examine the behaviour of the solution $e(t)$ of the non-linear controlled dynamical system

$$\dot{e}_x = P(e_x^{r-2}, t, \xi)e_x + Q(e_x^{r-1}, t, \xi)e_u, \quad e_x(t_0) = e_{x0}, \quad t \ge t_0 \tag{4.1}$$

(where e_x is the state vector (2.2) of the system and P and Q are the matrix functions (2.8)–(2.11) and (2.12)), which is closed by the control law e_u (1.29), (1.28). When relations (2.1)–(2.5) are taken into account, this control law can be represented in the form

$$e_u = \Gamma_0 e = e_{ux} \equiv \Gamma_0 R e_x = \bar{\Gamma}_0 e_x \tag{4.2}$$

where $\bar{\Gamma}_0$ is a constant $m \times n$ matrix of the form

$$\bar{\Gamma}_0 = \Gamma_0 R = \|\bar{\Gamma}_{01}, \dots, \bar{\Gamma}_{0r}\| \tag{4.3}$$

which consists of the $m \times m$ blocks

$$\bar{\Gamma}_{0k} = \Gamma_{0k} - \Gamma_{0,k+1}S_{k+1,k}, \quad k = 1, \dots, r-1; \quad \bar{\Gamma}_{0r} = \Gamma_{0r} \tag{4.4}$$

and an equation of the transients (in the closed system indicated) of the form

$$\dot{e}_x = \Gamma(e_x^{r-1}, t, \xi)e_x, \quad e_x(t_0) = e_{x0}, \quad t \ge t_0 \tag{4.5}$$

Here

$$\Gamma(e_x^{r-1}, t, \xi) = P(e_x^{r-2}, t, \xi) + Q(e_x^{r-1}, t, \xi)\bar{\Gamma}_0 = P(e_x^{r-2}, t, \xi) + P_{x1}(e_x^{r-1}, t, \xi)$$
$$= \|\Gamma_{kl}\|_{k, l = 1, \dots, r} \tag{4.6}$$

is an $n \times n$ matrix function that consists of the $m \times m$ blocks Γ_{kl} ($k, l = 1, \dots r$); P and Q are the matrix functions (2.8)–(2.11) and (2.12); when relations (2.12) and (1.16) are taken into account,

$$P_{x1}(e_x^{r-1}, t, \xi) = Q(e_x^{r-1}, t, \xi)\bar{\Gamma}_0 = Q_0(\sigma^{r-1}(e_x^{r-1}), t, \xi)\bar{\Gamma}_0$$
$$= \left\| \begin{matrix} 0 \\ P_{0r, r+1}(\sigma^{r-1}(e_x^{r-1}), t, \xi)\bar{\Gamma}_0 \end{matrix} \right\| \tag{4.7}$$

is an $m \times m$ partitioned matrix function.

Lemma 1. *Let the following conditions hold:*

1) *the matrix* $\bar{\Gamma}_0$ (4.3), (4.4) *has the form*

$$\bar{\Gamma}_0 = \|0, \bar{\Gamma}_{0r}\| = \|0, S_{r+1, r}\| \; (\bar{\Gamma}_{0r} = S_{r+1, r}) \tag{4.8}$$

2) *the* $S_{k+1,\, k}$ ($k = 1, \dots r$) *are non-singular constant* $m \times m$ *blocks, which can be represented in the form*

$$S_{k+1, k} = \gamma_{S, k+1, k} I_m, \quad k = 1, \dots, r \tag{4.9}$$

where the $\gamma_{S,\, k+1,\, k}$ ($k = 1, \dots r$) *are certain real numbers, which satisfy the inequalities*

$$\gamma_{S, k+1, k} > 0, \quad k = 1, \dots, r-1; \quad \gamma_{S, r+1, r} < 0, \quad \gamma_{S, 2, 1} > \gamma_{0S, 2, 1} = \Lambda_1[\bar{k}_{B1} + (r-1)]$$
$$|\gamma_{S, k+1, k}| > \gamma_{0S, k+1, k} = \Lambda_k\left[\bar{k}_{Bk} + \bar{\beta}_{Gkk} + (r-k) + \sum_{l=1}^{k-1} \alpha_{Gkl}\right], \quad k = 2, \dots, r \tag{4.10}$$
$$\Lambda_k = [2\underline{\lambda}(A_k)\underline{\lambda}^2(B_k)]^{-1}, \quad k = 1, \dots, r$$

$\underline{\lambda}(A_k), \underline{\lambda}(B_k)$ $(1, \dots, r)$ are real numbers, such that

$$0 < \underline{\lambda}(A_1) = \min_i \inf_{t \ge t_0, \xi \in \Omega_\xi} \lambda_i(A_1(t, \xi))$$
$$0 < \underline{\lambda}(A_k) = \min_i \inf_{(e_x^{k-1}, t, \xi) \in \Omega_{1x, k-1}} \lambda_i(A_k(\sigma^{k-1}(e_x^{k-1}), t, \xi)), \quad k = 2, \dots, r$$
$$0 < \underline{\lambda}(B_k) = \min_i \inf_{t \ge t_0, \xi \in \Omega_\xi} \lambda_i(B_k(t, \xi)), \quad k = 1, \dots, r$$
$$i = 1, \dots, m \tag{4.11}$$

the $\lambda_i(A_1(t,\xi))$, $\lambda_i(A_k(\sigma^{k-1}(e_x^{k-1})$ and $t,\xi))$, $\lambda_i(B_k(t,\xi))(i=1,\ldots,m;k=1,\ldots,r)$ *are eigenvalues of the matrix functions* A_k (1.18), (1.20) *and* B_k (1.19), (1.21) ($k=1,\ldots,r$), *respectively, the set is*

$$\Omega_{1x,k-1} = \{(e_x^{k-1},t,\xi) : e_x^{k-1} \in R^{m(k-1)},\ t \ge t_0,\ \xi \in \Omega_\xi\}$$

the $\bar{\beta}_{Gkk}$ and α_{Gkl} *are non-negative real numbers:*

$$\bar{\beta}_{G22} = \sup_{t \ge t_0,\ \xi \in \Omega_\xi} |\bar{G}_{22}(t,\xi)|, \quad \bar{\beta}_{Gkk} = \sup_{(e_x^{k-2},t,\xi) \in \Omega_{1x,k-2}} |\bar{G}_{kk}(e_x^{k-2},t,\xi)|, \quad k = 3,\ldots,r$$

$$\bar{G}_{kk}(e_x^{k-2},t,\xi) = B_k(t,\xi)[S_{k,k-1}P_{0,k-1,k}(\sigma^{k-2}(e_x^{k-2}),t,\xi)]$$

$$+[S_{k,k-1}P_{0,k-1,k}(\sigma^{k-2}(e_x^{k-2}),t,\xi)]^*B_k(t,\xi), \quad k = 2,\ldots,r$$

$$P_{012}(\sigma^0(e_x^0),t,\xi) \equiv P_{012}(t,\xi), \quad \bar{G}_{22}(e_x^0,t,\xi) \equiv \bar{G}_{22}(t,\xi)$$

$$\alpha_{Gk1} = \sup_{t \ge t_0,\ \xi \in \Omega_\xi} |G_{k1}(t,\xi)|^2, \quad k = 2,\ldots,r$$

$$\alpha_{Gkl} = \sup_{(e_x^{l-1},t,\xi) \in \Omega_{1x,l-1}} |G_{kl}(e_x^{l-1},t,\xi)|^2, \quad k = 3,\ldots,r; \quad l = 2,\ldots,k-1$$

$$G_{k1} \equiv G_{k1}(t,\xi) = B_k(t,\xi)\Gamma_{k1}(t,\xi), \quad k = 3,\ldots,r; \quad G_{k1}(e_x^0,t,\xi) \equiv G_{k1}(t,\xi)$$

$$G_{k,k-1}(e_x^{k-2},t,\xi) = B_k(t,\xi)\Gamma_{k,k-1}(e_x^{k-2},t,\xi) + \Gamma^*_{k-1,k}(e_x^{k-2},t,\xi)B_{k-1}(t,\xi), \quad k = 2,\ldots,r$$

$$G_{kl}(e_x^{l-1},t,\xi) = B_k(t,\xi)\Gamma_{kl}(e_x^{l-1},t,\xi), \quad k = 4,\ldots,r; \quad l = 2,\ldots,k-2 \tag{4.12}$$

Then the equilibrium position $e_x=0$ of the non-linear controlled dynamical system (4.1), (2.2), (2.8)–(2.12) closed by the control law e_u (4.2)–(4.4), (4.8)–(4.12) with linear feedback with respect to the state e_x is stabilizable, so that the following assertions hold:

1) the equilibrium position $e_x=0$ of the equation of the transients (in the closed system indicated) (4.5)–(4.12), (1.17)–(1.22) is asymptotically Lyapunov stable as a whole;
2) the solution $e_x(t)$ of this system satisfies the estimate

$$|e_x(t)| \le \beta_{x0}\exp[-\alpha_0(t-t_0)]|e_x(t_0)|, \quad t \ge t_0 \tag{4.13}$$

where β_{x0} and α_0 are positive real numbers:

$$\beta_{x0} = [\bar{\lambda}(B)\underline{\lambda}^{-1}(B)]^{1/2}$$

$$\underline{\lambda}(B) = \min_k\{\underline{\lambda}(B_k)\}, \quad \bar{\lambda}(B) = \max_k\{\bar{\lambda}(B_k)\}, \quad \bar{\lambda}(B_k) = \max_i \sup_{t \ge t_0,\ \xi \in \Omega_\xi} \lambda_i(B_k(t,\xi))$$

$$i = 1,\ldots,m; \quad k = 1,\ldots,r$$

$$\alpha_0 = \bar{\alpha}_0\bar{\lambda}^{-1}(B), \quad \bar{\alpha}_0 = \min_k \alpha_{exk}, \quad k = 1,\ldots,r$$

$$\alpha_{ex1} = \frac{1}{2}[\alpha_{G11} - (r-1)] > 0$$

$$\alpha_{exk} = \frac{1}{2}\left[\alpha_{Gkk} - (r-k) - \sum_{l=1}^{k-1}\alpha_{Gkl}\right] > 0, \quad k = 2,\ldots,r \tag{4.14}$$

$$\alpha_{G11} = 2\gamma_{S,2,1}\underline{\lambda}(A_1)\underline{\lambda}^2(B_1) - \bar{k}_{B1} > 0$$

$$\alpha_{Gkk} = 2|\gamma_{S,k+1,k}|\underline{\lambda}(A_k)\underline{\lambda}^2(B_k) - \bar{k}_{Bk} - \bar{\beta}_{Gkk} > 0, \quad k = 2,\ldots,r$$

Proof. We first note that in the equation of the transients (4.5)–(4.7) Γ is the partitioned matrix function (4.6), in which the matrix functions P, Q and $\bar{\Gamma}_0$ have the forms (2.8)–(2.11), (2.12) and (4.8), respectively, and the matrix function P_{x1} (4.7) has the form

$$P_{x1}(e_x^{r-1}, t, \xi) = \operatorname{diag}(0, P_{x1rr}(e_x^{r-1}, t, \xi)) \tag{4.15}$$

where

$$P_{x1rr}(e_x^{r-1}, t, \xi) = P_{0r,r+1}(\sigma^{r-1}(e_x^{r-1}), t, \xi)\bar{\Gamma}_{0r} = P_{0r,r+1}(\sigma^{r-1}(e_x^{r-1}), t, \xi)S_{r+1,r} \tag{4.16}$$

is an $m \times m$ block, $\sigma^k \equiv \sigma^k(e_x^k)$ is the mk-dimensional vector function (2.10) and, according to (4.8),

$$\bar{\Gamma}_{0r} = S_{r+1,r}$$

Now, let us consider the Lyapunov function

$$V(e_x, t, \xi) = e_x^* B(t, \xi) e_x \tag{4.17}$$

where

$$B(t, \xi) = \operatorname{diag}(B_1(t, \xi), \ldots, B_r(t, \xi)) = B^*(t, \xi) > 0, \quad t \ge t_0, \quad \xi \in \Omega_\xi \tag{4.18}$$

is a block-diagonal, symmetric, positive-definite matrix function, the B_k ($k = 1, \ldots, r$) are $m \times m$ blocks of form (1.19), (1.21), and stipulate that it satisfies the estimate

$$\underline{\lambda}(B)|e_x|^2 \le V(e_x, t, \xi) \le \bar{\lambda}(B)|e_x|^2, \quad (e_x, t, \xi) \in \Omega_{1x,r} \tag{4.19}$$

where $\underline{\lambda}(B) > 0$ and $\bar{\lambda}(B) > 0$ are constants defined by the second and third formulae in (4.14).

We calculate the derivative of the function $V(e_x(t), t, \xi)$ (4.17)–(4.19) with respect to the time t by virtue of the equation of the transients (4.5)–(4.12), (4.15), (4.16), (1.17)–(1.22), taking relations (4.8), into account (from the first condition of the lemma) for the matrix $\bar{\Gamma}_0$ and relations (4.9)–(4.12) (from the second condition of the lemma) for the blocks $S_{k+1,k}$ ($k = 1, \ldots, r$). We finally obtain

$$\dot{V}(e_x(t), t, \xi) = W(e_x(t), t, \xi), \quad t \ge t_0 \tag{4.20}$$

Here

$$W(e_x, t, \xi) = e_x^* G(e_x^{r-1}, t, \xi) e_x \tag{4.21}$$

is a quadratic form, where

$$\begin{gathered} G(e_x^{r-1}, t, \xi) = \dot{B}(t, \xi) + B(t, \xi)\Gamma(e_x^{r-1}, t, \xi) + \Gamma^*(e_x^{r-1}, t, \xi)B(t, \xi) = \|G_{kl}\|_{k,l=1,\ldots,r} \\ (G(e_x^{r-1}, t, \xi) = G^*(e_x^{r-1}, t, \xi),\ G^*(e_x^{r-1}, t, \xi) = \|(G^*)_{kl}\|_{k,l=1,\ldots,r},\ G_{kl} = (G^*)_{kl} = G_{lk}^*,\ k, l = 1, \ldots, r) \end{gathered} \tag{4.22}$$

is a symmetric matrix function of order $n \times n$, in which

$$\begin{gathered} G_{kl} = B_k\Gamma_{kl} + \Gamma_{lk}^* B_l = G_{lk}^* = (G^*)_{kl}, \quad k, l = 1, \ldots, r;\ k \ne l \\ G_{kk} = \dot{B}_k + B_k\Gamma_{kk} + \Gamma_{kk}^* B_k = (G^*)_{kk} = G_{kk}^*, \quad k = 1, \ldots, r \end{gathered} \tag{4.23}$$

is an $m \times m$ partitioned matrix function, where the B_k ($k = 1, \ldots, r$) are the $m \times m$ blocks (1.19), (1.21) and the Γ_{kl} ($k, l = 1, \ldots, r$) are the $m \times m$ partitioned matrices Γ (4.6).

We will estimate the quadratic form $W(e_x, t, \xi)$ (4.21)–(4.23).

For this purpose, we first estimate the quadratic forms

$$W_{11}(e_{x1}, t, \xi) = e_{x1}^* G_{11}(t, \xi) e_{x1} \tag{4.24}$$

$$W_{kk}(e_x^k, t, \xi) = e_{xk}^* G_{kk}(e_x^{k-1}, t, \xi) e_{xk}, \quad k = 2, \ldots, r \tag{4.25}$$

Taking into account relations (4.24), (4.25), (4.16), (4.12) and (4.14) and using the estimates

$$\begin{gathered} \tilde{W}_{kk}(e_x^k, t, \xi) = -2|\gamma_{S,k+1,k}| e_{xk}^*[B_k(t, \xi)A_k(\sigma^{k-1}(e_x^{k-1}), t, \xi)B_k(t, \xi)]e_k \le \\ \le -2|\gamma_{S,k+1,k}|\underline{\lambda}(A_k)\underline{\lambda}^2(B_k)|e_{xk}|^2, \quad k = 2, \ldots, r \end{gathered}$$

we obtain

$$W_{11}(e_{x1}, t, \xi) = e_{x1}^* G_{11}(t, \xi) e_{x1} = e_{x1}^* \dot{B}_1(t, \xi) e_{x1} - 2\gamma_{S,2,1} e_{x1}^* [B_1(t, \xi) A_1(t, \xi) B_1(t, \xi)] e_{x1} \le$$
$$\le \bar{k}_{B1}|e_{x1}|^2 - 2\gamma_{S,2,1} \underline{\lambda}(A_1)\underline{\lambda}^2(B_1)|e_{x1}|^2 = -\alpha_{G11}|e_{x1}|^2$$

$$W_{kk}(e_x^k, t, \xi) = e_{xk}^* G_{kk}(e_x^{k-1}, t, \xi) e_{xk} =$$
$$= e_{xk}^*[\dot{B}_k(t, \xi) + \bar{G}_{kk}(e_x^{k-2}, t, \xi) - 2|\gamma_{S,k+1,k}| B_k(t, \xi) A_k(\sigma^{k-1}(e_x^{k-1}), t, \xi) B_k(t, \xi)] e_{xk} =$$
$$= e_{xk}^* \dot{B}_k(t, \xi) e_{xk} + e_{xk}^* \bar{G}_{kk}(e_x^{k-2}, t, \xi) e_{xk} + \tilde{W}_{kk}(e_x^k, t, \xi) \le$$
$$\le \bar{k}_{Bk}|e_{xk}|^2 + \bar{\beta}_{Gkk}|e_{xk}|^2 - 2|\gamma_{S,k+1,k}| \underline{\lambda}(A_k)\underline{\lambda}^2(B_k)|e_{xk}|^2 = -\alpha_{Gkk}|e_{xk}|^2, \quad k = 2, \dots, r \tag{4.26}$$

where $\alpha_{Gkk} > 0$ $(k, 1, \dots r)$ are real numbers defined by the last r formulae in (4.14).

Next, using (4.20)–(4.26) and the inequalities

$$2[e_{xk}^* G_{kl}(e_x^{l-1}, t, \xi)] e_{xl} \le 2[|e_{xk}^*| |G_{kl}(e_x^{l-1}, t, \xi)|] |e_{xl}| \le$$
$$\le |e_{xl}|^2 + |G_{kl}(e_x^{l-1}, t, \xi)|^2 |e_{xk}|^2 \le |e_{xl}|^2 + \alpha_{Gkl}|e_{xk}|^2, \quad k = 2, \dots, r; \quad l = 1, \dots, k-1$$

where the $\alpha_{Gkl} > 0$ are real numbers given by (4.12), we estimate the quadratic form $W(e_x, t, \xi)$ (4.21)–(4.23). We obtain

$$W(e_x, t, \xi) = e_x^* G(e_x^{r-1}, t, \xi) e_x = \sum_{k=1}^{r} e_{xk}^* G_{kk}(e_x^{k-1}, t, \xi) e_{xk} +$$
$$+ 2\sum_{k=2}^{r}\left[\sum_{l=1}^{k-1} e_{xk}^* G_{kl}(e_x^{l-1}, t, \xi) e_{xl}\right] \le \sum_{k=1}^{r} e_{xk}^* G_{kk}(e_x^{k-1}, t, \xi) e_{xk} +$$
$$+ \sum_{k=2}^{r}\left\{\sum_{l=1}^{k-1}[|e_{xl}|^2 + \alpha_{Gkl}|e_{xk}|^2]\right\} = e_{x1}^*[G_{11}(t, \xi) + (r-1)I_m] e_{x1} +$$
$$+ \sum_{k=2}^{r} e_{xk}^*\left\{G_{kk}(e_x^{k-1}, t, \xi) + \left[(r-k) + \sum_{l=1}^{k-1}\alpha_{Gkl}\right] I_m\right\} e_{xk} \le$$
$$\le [-\alpha_{G11} + (r-1)]|e_{x1}|^2 + \sum_{k=2}^{r}\left[-\alpha_{Gkk} + (r-k) + \sum_{l=1}^{k-1}\alpha_{Gkl}\right]|e_{xk}|^2 =$$
$$= -2\sum_{k=1}^{r}\alpha_{exk}|e_{xk}|^2 \le -2\bar{\alpha}_0|e_x|^2 \le -2\alpha_0 V(e_x, t, \xi), \quad t \ge t_0 \tag{4.27}$$

where $\alpha_{exk} > 0$ $(k, 1, \dots r)$, $\bar{\alpha}_0 > 0$ and $\alpha_0 > 0$ are real numbers from (4.14) and (4.9)–(4.12).

Relations (4.20) and (4.27) lead to the estimate

$$\dot{V}(e_x(t), t, \xi) = W(e_x(t), t, \xi) \le -2\alpha_0 V(e_x(t), t, \xi), \quad t \ge t_0 \tag{4.28}$$

from which we find

$$V(e_x(t), t, \xi) \le V(e_x(t_0), t_0, \xi)\exp[-2\alpha_0(t - t_0)], \quad t \ge t_0$$

Hence, using relations (4.17)–(4.19) again, we obtain

$$|e_x(t)|^2 \le \beta_{x0}^2 |e_x(t_0)|^2 \exp[-2\alpha_0(t - t_0)], \quad t \ge t_0$$

where $\beta_{x0} > 0$ is a real number given by the first formula in (4.14). Therefore, the equilibrium position $e_x = 0$ of the equation of the transients, i.e., system (4.5)–(4.12), (1.17)–(1.22) is asymptotically Lyapunov stable as a whole with an estimate for the solution $e_x(t)$ of the form

$$|e_x(t)| \le \beta_{x0}|e_x(t_0)|\exp[-\alpha_0(t - t_0)], \quad e_x(t_0) = e_{x0}, \quad t \ge t_0 \tag{4.29}$$

i.e., the equilibrium position $e_u = 0$ of the non-linear controlled dynamical system (4.1), (2.8)–(2.12), (4.3) closed by the control law e_u (4.2)–(4.4), (4.8)–(4.12) with linear feedback with respect to the state e_x is stabilizable.

2°. We will examine the behaviour of the solution $e_x(t)$ of the non-linear controlled dynamical system of special form (2.6)–(2.16)

$$\dot{e}_x = P(e_x^{r-2}, t, \xi) e_x + Q(e_x^{r-1}, t, \xi) e_u + g_{ex}(e_x, t, e_\xi), \quad e_x(t_0) = e_{x0}, \quad t \ge t_0 \tag{4.30}$$

(where e_x is the state vector (2.2) of the system, P and Q are the matrix functions (2.8)–(2.11) and (2.12), and g_{ex} is the vector function (2.13)–(2.16)), which is closed by the control law e_u (1.29), (1.28) with linear feedback with respect to the state e_x. When relations (2.1)–(2.5) are taken into account, this control law can be represented in the form (4.2), i.e.,

$$e_u = e_{ux} = \bar{\Gamma}_0 e_x \tag{4.31}$$

where $\bar{\Gamma}_0$ is the constant matrix (4.3), (4.4), and the equation of the transiens (in the closed system indicated) of the form

$$\dot{e}_x = \Gamma(e_x^{r-1}, t, \xi)e_x + g_{ex}(e_x, t, e_\xi), \quad e_x(t_0) = e_{x0}, \quad t \geq t_0 \tag{4.32}$$

Here Γ denotes the matrix functions (4.6), (2.8)–(2.11) and (4.7), and g_{ex} is the vector function (2.13)–(2.16).

Lemma 2. *Let the conditions of* Lemma 1 *hold, and let the coefficients* k_{gexj} ($j = 0, 1, 2$) (2.16) *in estimate* (2.14) *for the vector function* g_{ex} (2.13) *satisfy the inequalities*

$$0 < k_{gex0} = |S|k_{ge0} < \infty, \quad 0 \leq k_{gex1} = |S||R|k_{ge1} < (1-\nu_0)\alpha_0\underline{\lambda}^{1/2}(B)\bar{\lambda}(B)^{-1/2}$$
$$0 < \nu_0 < 1, \quad 0 < k_{gex2} = |S||R|^2 k_{ge2} < \infty$$
$$[k_{gex1} - (1-\nu_0)\alpha_0\underline{\lambda}^{1/2}(B)\bar{\lambda}(B)^{-1/2}]^2 - 4k_{gex2}k_{gex0} > 0 \tag{4.33}$$

where the k_{gej} ($j = 0, 1, 2$) *and* α_0 *are constants that can be determined from relations* (1.24)–(1.26) *and* (4.14), (4.9)–(4.12).

Then the non-linear controlled dynamical system of special form (4.30), (2.2), (2.8)–(2.16), closed by the control law e_u (4.31), (4.3), (4.4), (4.8)–(4.12) with linear feedback with respect to the state e_x, is stabilizable. Thus, the following assertions hold for the solution $e_x(t)$ of the equation of the transients (in the closed non-linear controlled dynamical system indicated), i.e., system (4.32), (4.6), (2.8)–(2.11), (4.7), (2.13)–(2.16), (4.33):

1) system (4.32), (4.6), (2.8)–(2.11), (4.7), (2.13)–(2.16), 4.33) is dissipative "in the large", where the regions

$$\Omega_{ex0} = \{e_x \in R^n : \upsilon(e_x, t, \xi) = [e_x^* B(t, \xi)e_x]^{1/2} < \rho_{\upsilon 0}, t \geq t_0, \xi \in \Omega_\xi\} \tag{4.34}$$

$$\Omega_{ex1} = \{e_x \in R^n : \upsilon(e_x, t, \xi) = [e_x^* B(t, \xi)e_x]^{1/2} < \rho_{\upsilon 1}, t \geq t_0, \xi \in \Omega_\xi\}, \Omega_{ex1} \subset \Omega_{ex0} (4.35) \tag{4.35}$$

where B is the matrix function (4.18), (4.19) and the real numbers

$$\rho_{\upsilon 0} = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \rho_{\upsilon 1} = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \quad \rho_{\upsilon 0} > \rho_{\upsilon 1} > 0$$
$$(a = \bar{\lambda}^{1/2}(B)\underline{\lambda}^{-1}(B)k_{gex2}, b = \bar{\lambda}^{1/2}(B)\underline{\lambda}^{-1/2}(B)[k_{gex1} - (1-\nu_0)\alpha_0\underline{\lambda}^{1/2}(B)\bar{\lambda}^{-1/2}(B)] < 0,$$
$$c = \bar{\lambda}^{1/2}(B)k_{gex0}) \tag{4.36}$$

are estimates of the region of attraction and the limit set of this system, respectively;

2) in the region

$$\Omega_{exD} = \Omega_{ex0} \backslash \Omega_{ex1} \tag{4.37}$$

where Ω_{ex0} and Ω_{ex1} are the sets (4.34) and (4.35), respectively, the solution $e_x(t)$ of system (4.32), (4.6), (2.8)–(2.11), (4.7), (2.13)–(2.16), (4.33) satisfies the estimate

$$|e_x(t)| \leq \beta_{x0} e^{-\gamma_0(t-t_0)}|e_x(t_0)|, \quad e_x(t_0) \in \Omega_{exD}, \quad t_0 \leq t \leq t_*; \quad \gamma_0 = \nu_0\alpha_0 \tag{4.38}$$

where $\beta_{x0} > 0$ is a positive number defined by the first formula in (4.14), and t_* is a certain time such that

$$e_x(t_*; e_{x0}, t_0) \in \Omega_{ex1}, \quad (e_x(t; e_x(t_*), t_*) \in \Omega_{ex1}, \quad \forall t \geq t_* \geq t_0 \tag{4.39}$$

Proof. We will show that the conditions of the auxiliary lemma hold for system (4.32), (4.6), (2.8)–(2.11), (4.7), (2.13)–(2.16), (4.33) written in the form of the system

$$\dot{e}_x = f(e_x, t) + g_{ex}(e_x, t), \quad e_x(t_0) = e_{x0}, \quad t \geq t_0 \tag{4.40}$$

where the vector functions are defined by the formulae

$$f(e_x, t) \equiv \Gamma(e_x^{r-1}, t, \xi)e_x \tag{4.41}$$

$$g_{ex}(e_x, t) \equiv g_{ex}(e_x, t, e_\xi) \tag{4.42}$$

and $g_{ex}(e_x, t, e_\xi)$ is a vector function of the form (2.13)–(2.16), (4.33).

Consider the Lyapunov function

$$\upsilon(e_x, t, \xi) = (V(e_x, t, \xi))^{1/2} = [e_x^* B(t, \xi)e_x]^{1/2} \tag{4.43}$$

where $V(e_x, t, e_\xi)$ is the function (4.17)–(4.19). The function $\upsilon(e_x, t, e_\xi)$ (4.43) satisfies conditions 1 and 2 of the auxiliary lemma, where

$$\varepsilon_{\upsilon 1} = \underline{\lambda}^{1/2}(B), \quad \varepsilon_{\upsilon 2} = \varepsilon_{\upsilon 3} = \bar{\lambda}^{1/2}(B) \tag{4.44}$$

$\lambda(B) > 0$ and $\bar{\lambda}(B) > 0$ are constants defined by the second and third formulae in (4.14). When relations (4.44) and (4.33) are taken into account, the coefficients k_{gexj} ($j = 0, 1, 2$) (2.16) in estimate (2.14) for the vector function g_{ex} (2.13) satisfy estimates (3.8), i.e.,

$$0 < k_{g0} < \infty, \quad 0 \le k_{g1} \le (1 - \nu_0)\alpha_0 \varepsilon_{\upsilon 1}\varepsilon_{\upsilon 3}^{-1} 0, \quad 0 < \nu_0 < 1$$

$$0 < k_{g2} < \infty, \quad [k_{g1} - (1 - \nu_0)\alpha_0 \varepsilon_{\upsilon 1}\varepsilon_{\upsilon 3}^{-1}]^2 - 4k_{g2}k_{g0} > 0$$

from condition 3 of the auxiliary lemma, where

$$k_{g0} \equiv k_{gex0} = |S|k_{ge0} > 0, \quad k_{g1} \equiv k_{gex1} = |S||R|k_{ge1} \ge 0, \quad k_{g2} \equiv k_{gex2} = |S||R|^2 k_{ge2} > 0$$

k_{gej} ($j = 0, 1\ 2$) and $\alpha_0 > 0$ are constants that can be determined from (1.24)–(1.26) and (4.14), (4.9)–(4.12) and the $\varepsilon_{\upsilon i}$ ($i = 1, 2, 3$) are constants defined by (4.44).

Since the conditions of Lemma 1 are satisfied, then, according to this lemma, the derivative of the function $V(e_x(t), t, \xi)$ (4.17)–(4.19) with respect to the time t by virtue of system (4.5)–(4.7), (2.8)–(2.11) written in the form of the system

$$\dot{e}_x = f(e_x, t), \quad e_x(t_0) = e_{x0}, \quad t \ge t_0 \tag{4.45}$$

where f is the vector function (4.41), satisfies relations (4.2)–(4.28), i.e.,

$$\dot{V}(e_x(t), t, \xi) = W(e_x(t), t, \xi) \le -2\alpha_0 V(e_x(t), t, \xi), \quad t \ge t_0 \tag{4.46}$$

where $W(e_x(t), t, \xi)$ is the function (4.21)–(4.23) and α_0 is a positive number defined in relations (4.14), (4.10)–(4.12).

Taking into account estimate (4.46), we calculate the derivative of the function $\upsilon(e_x(t), t, \xi)$ (4.43), (4.17)–(4.19) with respect to the time t by virtue of system (4.45). We obtain

$$\begin{aligned}\dot{\upsilon}(e_x(t), t, \xi) &= [(V(e_x(t), t, \xi))^{1/2}]^{\cdot} = \frac{\dot{V}(e_x(t), t, \xi)}{2(V(e_x(t), t, \xi))^{1/2}} = \frac{1}{2\upsilon(e_x(t), t, \xi)}\dot{V}(e_x(t), t, \xi) = \\ &= \frac{1}{2\upsilon(e_x(t), t, \xi)}\frac{\partial V(e_x(t), t, \xi)}{\partial e_x} f(e_x, t) \le -\frac{1}{2\upsilon(e_x(t), t, \xi)} 2\alpha_0 V(e_x(t), t, \xi) = \\ &= -\frac{1}{2\upsilon(e_x(t), t, \xi)} 2\alpha_0 [\upsilon(e_x(t), t, \xi)]^2 = -\alpha_0 \upsilon(e_x(t), t, \xi), \quad t \ge t_0\end{aligned} \tag{4.47}$$

and the fourth condition of the auxiliary lemma is therefore satisfied.

Thus, the equation of the transients (4.32), (4.6), (2.8)–(2.11), (4.7), (2.13)–(2.16), (4.33) satisfies the conditions of the auxiliary lemma. Therefore, the assertions of this lemma, which are identical to the assertions of Lemma 2 for system (4.32), (4.6), (2.8)–(2.11), (4.7), (2.13)–(2.16), (4.33) written in the form of system (4.40)–(4.42), (2.13)–(2.16), (4.33) when relations (4.40)–(4.42) are taken into account, hold.

3°. Now we will examine the behaviour of the solution $e(t)$ of the non-linear controlled dynamical system of canonical form

$$\dot{e} = P_0(e^{r-2}, t, \xi)e + Q_0(e^{r-1}, t, \xi)e_u, \quad e(t_0) = e_0, \quad t \ge t_0 \tag{4.48}$$

(where P_0 and Q_0 are the matrix functions (1.15) and (1.16)), which is closed by the control law e_u (1.29), (1.28), and the equation of the transients (in the closed system indicated)

$$\dot{e} = \Gamma_e(e^{r-1}, t, \xi)e, \quad e(t_0) = e_0, \quad t \ge t_0 \tag{4.49}$$

Here

$$\Gamma_e(e^{r-1}, t, \xi) = P_0(e^{r-2}, t, \xi) + Q_0(e^{r-1}, t, \xi)\Gamma_0 = P_0(e^{r-2}, t, \xi) + P_1(e^{r-1}, t, \xi) \tag{4.50}$$

is an $n \times n$ matrix function, where P_0 is the $n \times n$ matrix function (1.15) and

$$P_1(e^{r-1}, t, \xi) = Q_0(e^{r-1}, t, \xi)\Gamma_0 = \left\| \begin{matrix} 0 \\ P_{0r, r+1}(e^{r-1}, t, \xi)\Gamma_0 \end{matrix} \right\| \tag{4.51}$$

is an $n \times n$ partitioned matrix function.

Theorem 1. *Let the matrix Γ_0 (1.28) of order $m \times n$ have the $m \times m$ blocks Γ_{0k} ($k = 1, \ldots, r$), which can be represented in the form*

$$\Gamma_{0k} = \Gamma_{0, k+1}S_{k+1, k} = S_{r+1, r}S_{r, r-1}\ldots S_{k+1, k}, \quad k = 1, \ldots, r-1; \quad \Gamma_{0r} \equiv S_{r+1, r} \tag{4.52}$$

and let the second condition of Lemma 1 *hold.*

Then the non-linear controlled dynamical system of canonical form (4.48), (1.14), (1.16) closed by the control law e_u (1.29), (1.28), (4.52), (4.9)–(4.12) with linear feedback with respect to the state e is stabilizable, so that for the solution $e(t)$ of the equation of the transient processes (in the closed non-linear controlled dynamical system indicated), i.e., system (4.49)–(4.52), (1.28), (4.9)–(4.12), assertions hold the following:

1) the equilibrium position $e = 0$ of system (4.49)–(4.52), (1.28), (4.9)–(4.12) is asymptotically Lyapunov stable as a whole;
2) the non-trivial solution $e(t)$ of system (4.49)–(4.52), (1.28), (4.9)–(4.12) satisfies the estimate

$$|e(t)| \le \beta_0 e^{-\alpha_0(t-t_0)}|e(t_0)|, \quad e(t_0) = e_0, \quad t \ge t_0 \tag{4.53}$$

where $\beta_0 = |R||S|\beta_{x0}$, β_{x0} and α_0 are positive numbers, defined in relations (4.14) and (4.9)–(4.12).

Proof. First, we transform the non-linear controlled dynamical system of canonical form (4.48), (1.15), (1.16) using the non-singular linear transformation of the coordinates of the state space

$$e_x = Se \; (e = S^{-1}e_x = Re_x)$$

of the form (2.1)–(2.5) into the non-linear controlled dynamical system (4.1), (2.8)–(2.12):

$$\dot{e}_x = P(e_x^{r-2}, t, \xi)e_x + Q(e_x^{r-1}, t, \xi)e_u, \quad e_x(t_0) = e_{x0}, \quad t \ge t_0$$

For the non-linear controlled dynamical system (4.1), (2.8)–(2.12) the control law e_u (4.31) has the matrix $\bar{\Gamma}_0$ (4.3), (4.4).When relations (4.52) are taken into account, this matrix has the $m \times m$ blocks

$$\bar{\Gamma}_{0k} = \Gamma_{0k} - \Gamma_{0,k+1}S_{k+1,k} = 0, \quad k = 1, \ldots, r-1; \quad \bar{\Gamma}_{0r} = \Gamma_{0r} \equiv S_{r+1,r} \tag{4.54}$$

and the first condition of Lemma 1 consequently holds. It follows from this and from fulfilment of the second condition of Lemma 1 that the assertions of Lemma 1 hold for this system.

It follows from the assertions of Lemma 1, the non-degeneracy of the linear replacement of variables of the form (2.1)–(2.5) and the estimates

$$|e| = |Re_x| \le |R||e_x|, \quad |e_x| = |Se| \le |S||e| \tag{4.55}$$

that the assertions of Theorem 1 hold for the non-linear controlled dynamical system of canonical form (4.48), (1.15), (1.16) closed by the control law e_u (1.29), (1.28), (4.52), (4.9)–(4.12) with linear feedback with respect to the state e, as well as for the solution $e(t)$ of the equation of the transients (in the closed non-linear controlled dynamical system indicated), i.e., system (4.48)–(4.51), (1.28), (4.52), (4.9)–(4.12).

Note that these assertions are similar to the assertions of Lemma 1 for the non-linear controlled dynamical system (4.1), (2.2), (2.8)–(2.12) closed by the control law e_u (4.2)–(4.4), (4.8)–(4.12) with linear feedback with respect to the state e_x and for the equation of the transients (4.5)–(4.12), (1.17)–(1.22) (in the closed system indicated). Theorem 1 is proved.

4°. In conclusion, we will examine the behaviour of the solution $e(t)$ of the original non-linear controlled dynamical system in deviations (1.13)–(1.26)

$$\dot{e} = P_0(e^{r-2}, t, \xi)e + Q_0(e^{r-1}, t, \xi)e_u + g_e(e, t, e_\xi), \quad e(t_0) = e_0, \quad t \ge t_0$$

closed by the control law e_u (1.29), (1.28) with linear feedback with respect to the state e, as well as the equation of the transients (in the closed system indicated)

$$\dot{e} = \Gamma_e(e^{r-1}, t, \xi)e + g_e(e, t, e_\xi), \quad e(t_0) = e_0, \quad t \ge t_0 \tag{4.56}$$

where Γ_e is the matrix function (4.50), (4.51) and g_e is the vector function (1.23)–(1.26).

Theorem 2. *Let the conditions of Theorem 1 hold, let the vector function g_e (1.23) satisfy estimate (1.24)–(1.26), and in estimate (2.14), (2.15) for the vector function g_{ex} (2.13) let the coefficients k_{gexj} ($j = 0, 1, 2$) (2.16) satisfy inequalities (4.33).*

Then the original non-linear controlled dynamical system in deviations (1.13)–(1.26), closed by the control law e_u (1.29), (1.28), (4.52), (4.9)–(4.12) with linear feedback with respect to the state e, is stabilizable so that for the solution $e(t)$ of the equation of the transients (in the closed non-linear controlled dynamical system indicated), i.e., system (4.56), (4.50), (4.51), (1.28), (4.52), (4.9)–(4.12), (1.23)–(1.26), (4.33), assertions hold the following:

1) system (4.56), (4.50), (4.51), (1.28), (4.52), (4.9)–(4.12), (1.23)–(1.26), (4.33) is dissipative "in the large", and the regions

$$\Omega_{e0} = \{e \in R^n: e = Re_x, e_x \in \Omega_{ex0}\} \tag{4.57}$$

$$\Omega_{e1} = \{e \in R^n : e = Re_x, e_x \in \Omega_{ex1}\}, \quad \Omega_{e1} \subset \Omega_{e0} \tag{4.58}$$

where Ω_{ex0} and Ω_{ex1} are the sets (4.34)–(4.36), are, respectively, estimates of the region of attraction and the limit set of this system; 2) in the region

$$\Omega_{eD} = \Omega_{e0} \backslash \Omega_{e1} \tag{4.59}$$

where Ω_{e0} and Ω_{e1} are sets (4.57) and (4.58), respectively, the solution $e(t)$ of system (4.56), (4.50), (4.51), (1.28), (4.52), (4.9)–(4.12), (1.23)–(1.26), (4.33) satisfies the estimate

$$|e(t)| \le \beta_0 e^{-\gamma_0(t-t_0)}|e(t_0)|, \quad e(t_0) \in \Omega_{eD}, \quad t_0 \le t \le t_*$$
$$\gamma_0 = \nu_0\alpha_0, \quad 0 < \nu_0 < 1, \quad \beta_0 = |R||S|\beta_{x0} \tag{4.60}$$

where β_{x0} and α_0 are positive numbers defined in relations (4.14), (4.9)–(4.12) and t_* is a certain time such that

$$e(t_*; e_0, t_0) \in \Omega_{e1}, \quad e(t; e_x(t_*), t_*) \in \Omega_{e1}, \quad \forall t \ge t_* \ge t_0 \tag{4.61}$$

Proof. First we transform the original non-linear controlled dynamical system in deviations (1.13)–(1.26) using the non-singular linear transformation of the coordinates of the state space (2.1)–(2.5) into the non-linear controlled dynamical system of special form (2.6)–(2.16):

$$\dot{e}_x = P(e_x^{r-2}, t, \xi)e_x + Q(e_x^{r-1}, t, \xi)e_u + g_{ex}(e_x, t, e_\xi), \quad e_x(t_0) = e_{x0}, \quad t \ge t_0$$

In estimate (2.14), (2.15) for the vector function g_{ex} (2.13), the coefficients k_{gexj} ($j = 0, 1, 2$) (2.16) satisfy inequalities (4.33).

For the non-linear controlled dynamical system of special form (2.6)–(2.16) (4.33), the control law e_u (4.31) has the matrix $\bar{\Gamma}_0$ (4.3), (4.4). When relations (4.52) are taken into account, this matrix consists of the $m \times m$ blocks $\bar{\Gamma}_{0k}$ ($k = 1, \ldots, r$) (4.54). It follows from this and from fulfilment of the second condition of Lemma 1 that the conditions of Lemma 2 hold and that the assertions of Lemma 1 are consequently valid for this system.

It follows from the assertions of Lemma 2, the non-degeneracy of the linear replacement of variables of the form (2.1)–(2.5) and estimates (4.55) that the assertions (similar to the assertions of Lemma 2) formulated in Theorem 2 also hold for the non-linear controlled dynamical system in deviations (1.13)–(1.26) closed by the control law e_u (1.29), (1.28), (4.52), (4.9)–(4.12) with linear feedback with respect to the state e, as well as for the solution $e(t)$ of the equation of the transients (in the closed non-linear controlled dynamical system indicated), i.e., system (4.56), (4.50), (4.51), (1.28), (4.52), (4.9)–(4.12), (1.13)–(1.26), (4.33). Theorem 2 is proved.

Remarks. In the control laws synthesized, viz., e_u (4.2)–(4.4), (4.8)–(4.12) for the non-linear controlled dynamical system (4.1), (2.2), (2.8)–(2.12), e_u (4.31), (4.3), (4.4), (4.8)–(4.12) for the non-linear controlled dynamical system of special form (4.30), (2.9), (2.8)–(2.16) and e_u (1.29), (1.28), (4.52), (4.9)–(4.12) for the non-linear controlled dynamical system of canonical form (4.48), (1.15), (1.16), as well as e_u (1.29), (1.28), (4.52), (4.9)–(4.12) for the original non-linear controlled dynamical system in deviations (1.13)–(1.26) (which were described above under the conditions of Lemmas 1 and 2 and Theorems 1 and 2, respectively), the corresponding formulae (4.8)–(4.12) and (4.52) for the elements of their matrices of the feedback loop gain $\bar{\Gamma}_{0k}$ ($k = 1, \ldots, r$) and Γ_{0k} ($k = 1, \ldots, r$) do not depend explicitly on the real parameter vector ξ of the non-linear controlled dynamical system, and depend only on the constants $\gamma_{S,\,k+1,\,k}$ ($k = 1, \ldots, r$) from estimates (4.10)–(4.12) (which are reducible under the conditions of the applicable lemmas and theorems indicated), which hold for all possible values from the assigned (admissible) set Ω_ξ.

5. Appendix

For a non-linear controlled dynamical system of the electromechanical type (for example, an electromechanical robotic manipulator[12]), which includes an actuating mechanism and electrical drive mechanisms based on dc motors with strong reduction gears, the dynamic equations have the form[12]

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial \Pi}{\partial q}\right]^* + Q_c \equiv A_0(q, \xi_0)\ddot{q} + b_0(q, \dot{q}, t, \xi_0) = Q_u$$
$$J\ddot{\alpha} + k_0\dot{\alpha} + i_p^{-1}\eta_p^{-1}Q_u = k_M I_a, \quad L\dot{I}_a + RI_a + k_e\dot{\alpha} = u \tag{5.1}$$

The first equation describes the dynamics of the actuating mechanism in the form of Lagrange's equations of the second kind, and the second and third equations describe the dynamics of the electric drive mechanisms. Here $q = \mathrm{col}(q_1, \ldots, q_m)$ is an m-dimensional vector of the generalized coordinates $q_1, \ldots, q_m$ of the mechanical part, i.e., the actuating mechanism, m is the number of degrees of freedom (mobility) of the actuating mechanism, ξ_0 and $\hat{\xi}_0$ are the real $p0$-dimensional parameter vector of the actuating mechanism and an estimate of it, i.e., its nominal value,

$$\xi_0 \in \Omega_{\xi 0}, \quad \hat{\xi}_0 \in \Omega_{\xi 0} \tag{5.2}$$

where $\Omega_{\xi 0}$ is a bounded closed set, and $A_0(q, \xi_0)$ is a continuously differentiable, symmetric, positive-definite $m \times m$ matrix function of the kinetic energy $T = \dot{q}^* A_0(q, \xi_0)\dot{q}/2$ of the actuating mechanism. Here

$$|A_0(q, \xi_0)| \le k_{A0}, \quad \forall q \in R^m, \quad \xi_0 \in \Omega_{\xi 0}; \quad 0 < k_{A0} < \infty \tag{5.3}$$

where k_{A0} is a certain constant. A similar estimate holds for the partial derivative of the matrix function A_0 with respect to the argument q:

$$b_0(q, \dot{q}, t, \xi_0) = \dot{A}_0(q, \xi_0)\dot{q} - \frac{1}{2}\left[\frac{\partial[\dot{q}^* A_0(q, \xi_0)\dot{q}]}{\partial q}\right]^* + Q_\Pi + Q_c \tag{5.4}$$

$$Q_\Pi \equiv Q_\Pi(q, \xi_0) = \left[\frac{\partial \Pi(q, \xi_0)}{\partial q}\right]^* = \operatorname{col}(Q_{\Pi 1}(q, \xi_0), \dots, Q_{\Pi m}(q, \xi_0))$$
$$Q_{\Pi i}(q, \xi_0) = \frac{\partial \Pi(q, \xi_0)}{\partial q_i}, \quad i = 1, \dots, m \tag{5.5}$$

$$Q_c \equiv Q_c(q, \dot{q}, t, \xi_0) = \Theta_c(q, t, \xi_0)\dot{q} \tag{5.6}$$

Q_Π is an m-dimensional vector of the potential forces acting on the actuating mechanism, $\Pi = \Pi(q, \xi_0)$ is the potential energy of the actuating mechanism, $Q_{\Pi i}(q, \xi_0)$ $(i = 1, \dots, m)$ are continuously differentiable functions, Q_c is an m-dimensional vector of generalized resistance forces (torques) acting on the degrees of mobility of the actuating mechanism, $\Theta_c(q, t, \xi_0)$ is a continuously differentiable $m \times m$ matrix function, I_a is an m-dimensional vector of the currents in the armature circuits of the dc motors, $u = \operatorname{col}(u_1, \dots, u_m)$ is an m-dimensional control vector, whose components control the voltages supplied to the armature circuits of the dc motors, $Q_u = \operatorname{col}(Q_{u1}, \dots, Q_{um})$ is an m-dimensional vector of the generalized forces (torques) that are applied to the degrees of mobility of the actuating mechanism, J, k_0, k_m, L, R and k_e are diagonal matrices of the electromechanical parameters of the dc motors, which are positive real quantities, i_p and η_p are diagonal matrices of the gear ratios and efficiencies of the reduction gears, and $\alpha = i_p q$ is an m-dimensional vector of the angles of rotation the motor shafts.

We use

$$\xi_1 \in \Omega_{\xi 1}, \quad \hat{\xi}_1 \in \Omega_{\xi 1} \tag{5.7}$$

to denote the $(p1 = 8m)$-dimensional parameter vector of the electric drive mechanisms and an estimate of it, i.e., its nominal value, whose components are diagonal elements of the matrices J, k_0, k_m, L, R, k_e, i_p, η_p and their estimates $\hat{J}$, $\hat{k}_0$, $\hat{k}_M$, $\hat{L}$, $\hat{R}$, $\hat{k}_e$, $\hat{i}_p$, $\hat{\eta}_p$ ($\Omega_{\xi 1}$ is a bounded closed set), and we use

$$\xi = \operatorname{col}(\xi_0, \xi_1) \in \Omega_\xi, \quad \hat{\xi} = \operatorname{col}(\hat{\xi}_0, \hat{\xi}_1) \in \Omega_\xi$$
$$(\Omega_{\xi 0} \cap \Omega_{\xi 1} = \varnothing, \ \ \Omega_\xi = \Omega_{\xi 0} \cup \Omega_{\xi 1}) \tag{5.8}$$

to denote the $(p = p0 + p1)$-dimensional parameter vector of the non-linear controlled dynamical system (5.1)–(5.7) and an estimate of it, i.e., its nominal value.

The dynamic equations of a non-linear controlled dynamical system of form (5.1)–(5.8)

$$e = \operatorname{col}(e_1, e_2, e_3), \quad z = \operatorname{col}(q, \dot{q}, I_a), \quad z_p = \operatorname{col}(q_p, \dot{q}_p, I_{ap}) \in \Omega_{zp}$$
$$e_1 = q - q_p, \quad e_2 = \dot{q} - \dot{q}_p, \quad e_3 = I_a - I_{ap}$$
$$\Omega_{zp} = \{z_p = \operatorname{col}(q_p, \dot{q}_p, I_{ap}) \in R^{3m} : |\dot{q}_p(t)| \le k_{z2p} < \infty, \ |I_{ap}(t)| \le k_{z3p} < \infty, \ t \ge t_0\} \tag{5.9}$$

which are written in the deviations e and e_u (1.8) from their programmed values $z_p = z_p(t) \equiv z_p(t, \hat{\xi})$ and $u_p = u_p(t) \equiv u_p(t, \hat{\xi})$ (where $k_{zip} \in [0, <\infty)$ ($i = 2, 3$) are certain constants), can be represented in the form of system (1.13)–(1.26), where $n = 3m$, $r = 3$, and

$$A_1(t, \xi) = B_1(t, \xi) = I_m, \quad A_2(e^1, t, \xi) = [Ji_p^2\eta_p + A_0(q, \xi_0)]^{-1}, \quad B_2(t, \xi) = i_p\eta_p k_M$$

$$A_3(e^2, t, \xi) = L^{-1}, \quad B_3(t, \xi) = I_m$$

$$g_{e1}(e^1, t, e_\xi) = 0, \quad g_{e2}(e^2, t, e_\xi) = \bar{g}_{e2}(e^2, t, e_\xi) + \hat{F}_{e2}(t, e_\xi)$$

$$\bar{g}_{e2}(e^2, t, \xi) = \Delta\bar{A}(e_1, t, \xi)[k_M I_{ap} - b(q_p, \dot{q}_p, t, \xi)] - A^{-1}(q, \xi)\Delta b(e_1, e_2, t, e_\xi)$$

$$\Delta\bar{A}(e_1, t, \xi) = A^{-1}(e_1 + q_p, \xi) - A^{-1}(q_p, \xi), \quad A(q, \xi) = i_p^{-1}\eta_p^{-1}[Ji_p^2\eta_p + A_0(q, \xi_0)]$$

$$b(q, \dot{q}, t, \xi) = k_0 i_p \dot{q} + i_p^{-1}\eta_p^{-1} b_0(q, \dot{q}, t, \xi)$$

$$\Delta b(e_1, e_2, t, \xi) = b(q, \dot{q}, t, \xi) - b(q_p, \dot{q}_p, t, \xi) = k_0 i_p e_2 + i_p^{-1}\eta_p^{-1}\Delta b_0(e_1, e_2, t, \xi_0)$$

$$\Delta b_0(e_1, e_2, t, \xi_0) = b_0(e_1 + q_p, e_2 + \dot{q}_p, t, \xi_0) - b_0(q_p, \dot{q}_p, t, \xi_0)$$

$$\hat{F}_{e2}(t, e_\xi) = A^{-1}(q_p, e_\xi + \hat{\xi})[k_M I_{ap} - b(q_p, \dot{q}_p, t, e_\xi + \hat{\xi})] - A^{-1}(q_p, \hat{\xi})[\hat{k}_M I_{ap} - b(q_p, \dot{q}_p, t, \hat{\xi})]$$

$$g_{e3}(e^3, t, e_\xi) = L^{-1}[-Re_3 - k_e i_p e_2 + (-L^{-1}R + \hat{L}^{-1}\hat{R})z_{p3}$$
$$+ (-L^{-1}k_e i_p + \hat{L}^{-1}\hat{k}_e\hat{i}_p)z_{p2} + (L^{-1} - \hat{L}^{-1})u_p]$$

$$L^{-1} = \Delta\bar{L} + \hat{L}^{-1}, \quad R = \Delta R + \hat{R}, \quad k_e = \Delta k_e + \hat{k}_e, \quad i_p = \Delta i_p + \hat{i}_p \tag{5.10}$$

Here the $e^i = \mathrm{col}(e_1, \ldots, e^i)$ ($i = 1, 2, 3$) are mi vectors, and

$$|\Delta\bar{A}(e_1, t, \xi)| \le \bar{k}_{\Delta A}|e_1|, \quad \forall e_1 \in R^m, \quad t \ge t_0, \quad \xi \in \Omega_\xi \tag{5.11}$$

where $\bar{k}_{\Delta A} \in [0, <\infty)$ is a certain constant; similar estimates hold for the partial derivatives of the matrix function $\Delta\bar{A}$ with respect to its arguments e_1 and t; $A_2(e^1, t, \xi)$ is a symmetric positive definite matrix function;

$$\Delta Q_\Pi(e_1, t, \xi_0) = Q_\Pi(e_1 + q_p, \xi_0) - Q_\Pi(q_p, \xi_0)$$

$$|\Delta Q_\Pi(e_1, t, \xi_0)| \le k_{\Delta Q\Pi}|e_1|, \quad \forall e_1 \in R^m, \quad t \ge t_0, \quad \xi_0 \in \Omega_{\xi 0} \tag{5.12}$$

$$\Delta Q_c(e_1, e_2, t, \xi_0) = Q_c(q, \dot{q}, t, \xi_0) - Q_c(q_p, \dot{q}_p, t, \xi_0)$$

$$= \Theta_c(q, t, \xi_0)\dot{q} - \Theta_c(q_P, t, \xi_0)\dot{q}_p = \Theta_c(e_1 + q_p, t, \xi_0)e_2 + \Delta\Theta_c(e_1, t, \xi_0)\dot{q}_p \tag{5.13}$$

$$\Delta\Theta_c(e_1, t, \xi_0) = \Theta_c(e_1 + q_p, t, \xi_0) - \Theta_c(q_p, t, \xi_0)$$

$$|\Delta Q_c(e_1, e_2, t, \xi_0)| \le k_{\Delta Qc1}|e_1| + k_{\Delta Qc2}|e_2|, \quad \forall e_1, e_2 \in R^m, \quad t \ge t_0, \quad \xi_0 \in \Omega_{\xi 0} \tag{5.14}$$

where $k_{\Delta Q\Pi} \ge 0$, $k_{\Delta Qc1} \ge 0$ and $k_{\Delta Qc2} > 0$ are certain constants.

It follows from (5.1)–(5.4) that the following estimates hold

$$|\Delta b(e_1, e_2, t, \xi)| \le k_{\Delta b1}|e_1| + k_{\Delta b2}|e_2| + k_{\Delta b22}|e_2|^2, \quad \forall e_1, e_2 \in R^m, \quad t \ge t_0, \quad \xi \in \Omega_\xi \tag{5.15}$$

$$|\Delta b_0(e_1, e_2, t, \xi_0)| \le k_{\Delta b01}|e_1| + k_{\Delta b02}|e_2| + k_{\Delta b022}|e_2|^2, \forall e_1, e_2 \in R^m, t \ge t_0, \xi_0 \in \Omega_{\xi 0} \tag{5.16}$$

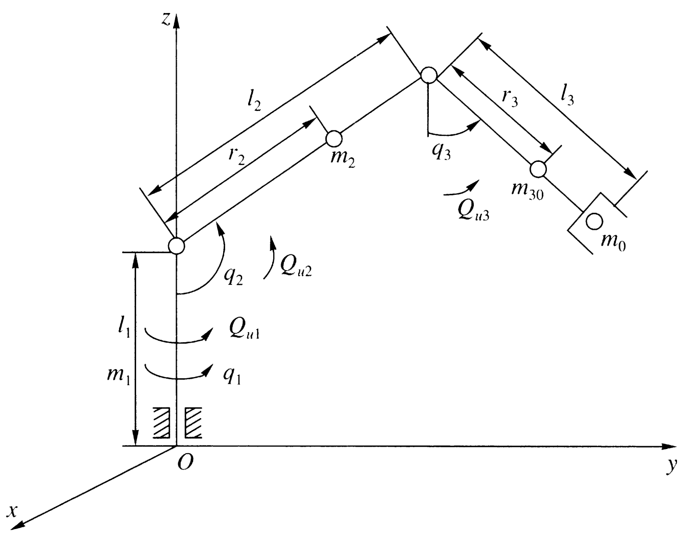
where $k_{\Delta bj} \ge 0$, $k_{\Delta b22} \ge 0$, $k_{\Delta b0j} \ge 0$ and $k_{\Delta b022} > 0$ ($j = 1, 2$) are certain constants.

It follows from relations (5.1)–(5.16) that relations (1.18)–(1.26) hold. Therefore, the non-linear controlled dynamical system (1.9), (5.1)–(5.16) is a non-linear controlled dynamical system of the form (1.13)–(1.26).

Example. As a an example of a non-linear controlled dynamical system, we will consider an electromechanical robotic manipulator with an actuating mechanism, i.e., a spatial manipulator with three degrees of freedom, whose kinematic diagram is shown in the figure. Here the q_i ($i = 1, 2, 3$) are generalized coordinates of the actuating mechanism of the manipulator, i.e., the angles formed by the corresponding links, which are degrees of mobility of the actuating mechanism with the axes of the stationary Cartesian system of coordinates $Oxyz$, l_i and m_i are the length of the ith link and its mass, r_1 is the radius of the shaft (a cylinder), i.e., the first link of the actuating mechanism, r_2 and r_3 are the distances from the centres of gravity of the second and third links (taking into account the mass m of the load in its gripper) to the axis of rotation the respective link, Q_{ui} is the load torque on the ith link of the actuating mechanism, and $m=3$ is the number of degrees of freedom (mobility) of the actuating mechanism.

In the dynamic equation (5.1), (5.8) of the actuating mechanism of such an electromechanical robotic manipulator, the symmetric, positive-definite 3×3 matrix function $A_0(q, \xi_0)$ of the kinetic energy of the actuating mechanism has the elements

$$a_{011}(q,\xi_0) = J_{01} + m_2 r_2^2 \sin^2 q_2 + m_{30}(l_2 \sin q_2 + r_3 \sin q_3)^2, \quad a_{01i} = a_{0i1} = 0, \quad i = 2, 3$$
$$a_{022}(\xi_0) = m_2 r_2^2 + m_3 l_2^2, \quad a_{023}(q,\xi_0) = a_{032}(q,\xi_0) = \frac{1}{2} m_{30} l_2 r_3 \cos(q_2 - q_3),$$
$$a_{033}(\xi_0) = m_{30} r_3^2 \tag{5.17}$$



Here

$$q = \mathrm{col}(q_1, q_2, q_3), \quad \xi_0 = \mathrm{col}(l_1, r_1, l_2, r_2, l_3, r_3, m_1, m_2, m_3, m_0, \tilde{g}, k_{BT1}, k_{BT2}, k_{BT3})$$
$$J_{01} = m_1 r_1^2/2, \quad m_{30} = m_3 + m_0 \tag{5.18}$$

(J_{01} is the moment of inertia of the first link of the actuating mechanism).

We write an expression for the potential energy of the actuating mechanism

$$\Pi(\tilde{q}, \xi_0) = m_2 \tilde{g} r_2(1-\cos q_2) + m_{30}\tilde{g}[l_2(1-\cos q_2) + r_3(1-\cos q_3)], \quad \tilde{q} = \mathrm{col}(q_2, q_3) \tag{5.19}$$

where $\tilde{g}$ is the acceleration due to gravity, and an expression for the three-dimensional vector of the resistive torques acting on the degrees of mobility of the actuating mechanism

$$Q_c(\dot{q}, \xi_0) = \mathrm{col}(k_{BT1}\dot{q}_1, k_{BT2}\dot{q}_2, k_{BT3}(-\dot{q}_2 + \dot{q}_3)) \tag{5.20}$$

where k_{BTi} ($i=1, 2, 3$) are damping (viscous friction) coefficients.

We will show that the non-linear controlled dynamical system (1.9), (5.1)–(5.8), (5.17)–(5.20), (5.9), (5.10), which describes the dynamics of the electromechanical robotic manipulator (5.1)–(5.8), (5.17)–(5.20) in deviations is a non-linear controlled dynamical system of the form (1.13)–(1.26).

In fact, it can be concluded that estimates (5.3), (5.11) and (5.13) hold for the matrix function $A_0(q, \xi_0)$ with elements (5.17) and for the vector function Q_c (5.20) from the dynamic equations of the actuating mechanism (5.1), (5.8), respectively.

Next, taking into account relations (5.19) and (5.5) for the potential energy of the actuating mechanism $\Pi(\tilde{q}, \xi_0)$ and the vector function $Q_\Pi(\tilde{e}_1, \xi_0)$, where $\tilde{e}_1 = \mathrm{col}(e_{12}, e_{13})$, we define the vector function

$$\Delta Q_\Pi(\tilde{e}_1, t, \xi_0) = Q_\Pi(\tilde{e}_1 + \tilde{q}_p, \xi_0) - Q_\Pi(\tilde{q}_p, \xi_0) = k_0(\xi_0)\eta(\tilde{e}_1, t), \quad \tilde{q}_p = \mathrm{col}(q_{2p}, q_{3p}) \tag{5.21}$$

where $k_0(\xi_0) = \| k_{01}(\xi_0), k_{02}(\xi_0) \|$ is a 3×2 matrix, $k_{01}(\xi_0) = \mathrm{col}(0, k_{012}(\xi_0), 0)$ and $k_{02}(\xi_0) = \mathrm{col}(0, 0, k_{023}(\xi_0))$ are three-dimensional vectors, $k_{012}(\xi_0) = (m_2 r_2 + m_{30} l_2)\tilde{g}$ and $k_{023}(\xi_0) = m_{30} r_3 \tilde{g}$, and the vector function

$$\eta(\tilde{e}_1, t) = \mathrm{col}(\eta_1(e_{12}, t), \eta_2(e_{13}, t))$$
$$(\eta_i(e_{1,i+1}, t) = \sin(e_{1,i+1} + q_{i+1,p}) - \sin(q_{i+1,p}), \ i = 1, 2) \tag{5.22}$$

We will show that the vector function $\Delta Q_\Pi(\tilde{e}_1, t.\xi_0)$ (5.21), (5.22) satisfies estimate (5.12).

Using the formula for finite increments of this vector function (Ref. 13, p. 122, Lemma 3.1), we estimate its absolute value:

$$|\Delta Q_\Pi(\tilde{e}_1, t, \xi_0)| = |k_0(\xi_0)\eta(\tilde{e}_1, t)| = |k_0(\xi_0)\chi\tilde{e}_1(t)| \le |k_0(\xi_0)||\chi||\tilde{e}_1(t)| \le \mu_0|\tilde{e}_1(t)|, \quad t \ge t_0 \tag{5.23}$$

Here

$$\chi = \int_0^1 J_\eta(s\tilde{e}_1(t), t)ds, \quad J_\eta(\tilde{e}_1, t) = \frac{\partial\eta(\tilde{e}_1, t)}{\partial\tilde{e}_1} = \mathrm{diag}(\cos(e_{12} + q_{2p}), \cos(e_{13} + q_{3p}))$$
$$\sup_{\sigma \in [0, \tilde{e}_1],\, t \geq t_0} |J_\eta(\sigma, t)| = \tilde{\mu}_0, \quad [0, \tilde{e}_1] = \{\nu : \nu = s\tilde{e}_1, 0 \leq s \leq 1\}$$
$$\tilde{\mu}_0 \leq \sqrt{2}, \quad \mu_0 = \tilde{k}_0\tilde{\mu}_0, \quad \tilde{k}_0 = \max_{\xi_0 \in \Omega_{\xi 0}} |k_0(\xi_0)| \tag{5.24}$$

It follows from relations (5.22)–(5.24) that estimate (5.12) holds for the vector function $\Delta Q_\Pi(\tilde{e}_1, t.\xi_0)$ (5.21).

Hence it follows that relations (1.18)–(1.26) hold. Therefore, the non-linear controlled dynamical system (1.9), (5.1)–(5.8), (5.17)–(5.20), (5.3), (5.11), (5.13), (5.21)–(5.22), which describes the dynamics of the electromechanical robotic manipulator (5.1)–(5.10), (5.17)–(5.20) in deviations is a non-linear controlled dynamical system of the form (1.13)–(1.26).

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